[Ghanim *et al.*, 5(7): July, 2018]

ISSN 2349-0292

Impact Factor 3.802

GLOBAL JOURNAL OF ADVANCED ENGINEERING TECHNOLOGIES AND SCIENCES

CONFIRMATION OF THE DE POLIGNAC AND THE TWIN PRIMES CONJECTURES

Mohammed Ghanim

* Ecole Nationale de Commerce et de Gestion B.P 1255 Tanger Maroc

DOI: 10.5281/zenodo.1309144

ABSTRACT

I show the veracity of the De Polignac conjecture, remained open since 1849, which says that for any natural integer k there is an infinite number of pairs of prime integers differing by 2k, and I deduce the twin prime conjecture (i.e. the case k=1) . I do this by using the Schoenfeld inequality that I have proved in [10] and by using the same approach that I have used in [11].

KEYWORDS: the De Polignac conjecture-the twin primes conjecture –the Schoenfeld inequality –the prime counting function-the rarefaction law of prime numbers- the Hospital rule 2010 Mathematics Subject classification: 11 A xx (Elementary Number theory)

INTRODUCTION

Definition1: We call "the De Polignac conjecture", the following assertion: "for any not null natural integer k, it exists an infinite number of pairs of prime integers (p,q) such that:$p - q = 2k$" .The case: $k = 1$ is the twin prime conjecture which says: "it exists an infinite number of prime integers p such that the numbers p+2 are also prime ".

History: this conjecture was announced by the French mathematician Alphonse De Polignac (1826-1862) in 1849 [4], [5] as a generalization of the twin prime conjecture.

From this date up to now the De Polignac conjecture (with the twin prime conjecture) remained without any rigorous proof.

in 1900, the German Mathematician David Hilbert (1862-1943) said in his conference delivered before the second international congress of mathematicians (hold at Paris in 1900) in the 8th point about the prime number problems : « After an exhaustive discussion of Riemann's prime number formula, perhaps we may sometime be in a position to attempt the rigorous solution on Goldbach problem… and further to attack the well known question whether there are an infinite number of pairs of prime numbers with the difference 2, or even the more general problem whether the linear Diophantine equation: $ax + by = c$ (with given integral coefficients each prime to the others), is always solvable in prime numbers x and y ?... » [14].

In 1922, the English Mathematicians G.H. Hardy (1877-1947) and John Edensor Littlewood (1885-1977) conjectured [13] that if:$\pi_d(x) = card\{p \in \mathbb{P}_x \text{such that}\, p + d \in \mathbb{P}_{x+d}\}$ (for $d\ integer \geq 2$) and if $c_2 = \prod_{p\in\mathbb{P},p\geq 3}\frac{p(p-2)}{(p-1)^2}$=0.66016…, with $R_d = \prod_{p\in\mathbb{P},p\geq 3,p|d}(\frac{p-1}{p-2})$ (Which is an irrational number≥ 1). Where $\mathbb{P}$ is the set of prime positive integers and $\mathbb{P}_x = \{p \in \mathbb{P}, such\ that\ p \leq x\}$, then:

$$\pi_d(x) \text{ Is equivalent, in the neighborhood of infinity, to: } 2c_2R_d\int_2^x \frac{dt}{(\ln(t))^2}$$

In 1940, the Hungarian mathematician Paul Erdos (1913-1996), showed in [6] that it exists $c < 1$ and it exists an infinite number of integers k such that:$p_{k+1} - p_k < c\ln(p_k)$, where $(p_k)_{k\geq 1}$ denotes the strictly increasing sequence of prime positive integers.

In 1966, the Chinese Mathematician Jingrun Chen (1933-1996) showed [2] (See also [19]) the existence of an infinitely many prime integers p such that p+2 is the product of at most two prime factors.

In 2003, the American mathematician Daniel Goldston (1954-) and the Turkish mathematician Cem Yildrim(1961-) showed that for any c>0, it exists an infinite number of integers k such that:$p_{k+1} - p_k < c\ln(p_k)$.

The proof containing some mistakes, the ultimo version of D.Goldston works on twin primes revised and corrected with the help of three colleagues: Turkish, Japanese and Hungarian, was online on June 8, 2005[16].

In 2013, the Chinese mathematician Zhang Yitang (1955-.) showed [18] a weak form of the twin prime conjecture, that is the existence of an infinite number of pairs of prime integers which does not differ one of the other more than 7.10^7.

Finally, there are online some sites giving lists of twin primes for higher values [1].
For recent references on the subject see [8] and [19].

The note : my purpose in present brief note is to show that for any not null natural integer k, it exists an infinite number of pairs (p,q) of prime integers such that : $p-q=2k$, conjectured, since 1849, by A. De Polignac. I use, for this, the Schoenfeld inequality that I have showed recently in [10] (i.e. the Riemann Hypothesis) and I use the same method as that I have developed in [11].

The paper is organized as follows: the §1 contains an introduction giving the necessary definition with some brief history, the §2 recalls the results needed in the proofs of our main theorems, the §3 gives the proof of the De Polignac conjecture, the §4 contains the deduction of the twin primes conjecture and finally the §5 contains the references of the paper for further reading.
Our main results are:

Theorem: (the De Polignac conjecture is true) $\forall k \in \mathbb{N}^* \exists$ an infinite number of pairs of prime integers (p,q) such that ; $p-q=2k$

Corollary: (the twin primes conjecture is true) $\exists$ an infinite number of pairs of prime integers (p,q) such that : $p-q=2$

INGREDIENTS OF THE PROOFS

We will need the below results in the proofs of our main theorems.

Proposition1: (See [17], [20]) Denoting, for a finite subset A of a set E, by $card(A)$ the number of its elements, we have:
(1)$card(\emptyset)=0$ and $card(\{a\})=1$
(2) If A *and* B are two finite subsets of E, then: $A \subset B \Rightarrow card(A) \leq card(B)$
(3)Let $A \times B = \{(a,b), a \in A, b \in B\}$ the Cartesian product of the sets A and B. If A, B are finite, we have: $card(A \times B) = card(A)card(B)$
(4) If $(A_i)_{1 \leq i \leq n}$ is a finite sequence of finite subsets of a set E such that $A_i \cap A_j = \emptyset$ for $i \neq j$ then :$card(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} card(A_i)$

Definition2: (See [22]) Recall that a prime integer p≥ 0 is an integer having its set of divisors= {1, p}. Let: $\mathbb{P} = \{p \in \mathbb{N}, p \text{ prime}\} = \{2,3,5,7,11,13,17,19,\ldots\}$ the set of positive prime integers. We know since Euclid (3[thd] Century before Jesus Christ) (See [7]) that $\mathbb{P}$ is an infinite strictly increasing sequence$(p_k)_{k\geq 1}$. For n an $integer \geq 2$, if $\mathbb{P}_n = \{p \in \mathbb{P}, p \leq n\}$ the natural number: $\pi(n) = card(\mathbb{P}_n)$, its number of elements, is called the prime counting function. We have: $\lim_{n\to+\infty} \pi(n) = +\infty$ and $n \leq m \Rightarrow \pi(n) \leq \pi(m)$

Definition3: (See [23]) two functions f *and* g defined on an interval of $\mathbb{R}$ containing a point a, are said to be equivalent on the neighborhood of a (we note $f \sim_a g$) if it exists a function ϵ defined on a neighborhood of a such that : $f(t) = \big(1+\epsilon(t)\big)g(t)$ *and* $\lim_{t\to a} \epsilon(t) = 0$

Two functions f *and* g defined on an interval$]b,+\infty[$ $(resp.]-\infty,c[)$ are said to be equivalent on the neighborhood of $+\infty$ $(resp. -\infty)$ if the functions $t \to f\left(\frac{1}{t}\right)$ and $t \to g(\frac{1}{t})$ are equivalent in the neighborhood of 0.
We know, by the rarefaction law of prime numbers showed independently by the Belgian mathematician Charles Jean Etienne Gustave De La vallée Poussin (1866-1962) [3] and the French mathematician Jacques Salomon Hadamard (1865-1963)[12], that :

Proposition2: (the rarefaction law of prime numbers) [3], [12] we have: $\lim_{k\to+\infty}\frac{\pi(k)\ln(k)}{k}=1$

I.e. the functions:$f(t)=\pi(t)\ and\ g(t)=\frac{t}{\ln(t)}$ are equivalent in the neighborhood de $+\infty$.

Definition4 : (See [23]) We have :

$$f=O(g)\ in\ the\ neighborhhod\ of\ +\infty \Leftrightarrow \exists A\in\mathbb{R}\exists B>0\ \forall t: t>A\Rightarrow |f(t)|<B|g(t)|$$

I have showed in [10] that:

Proposition3: (The Schoenfeld inequality [10]) we have:

$\forall n\geq 2657\ \ |\pi(n)-\int_0^n\frac{dt}{\ln(t)}|\leq\frac{\sqrt{n}\ln(n)}{8\pi}$

i.e.: $\pi(n)=\int_0^n\frac{dt}{\ln(t)}+O(\sqrt{n}\ln(n))$ for $n\geq 2657$

The following rule, allowing in practice the determination of many limits, was discovered by the French Mathematician Guillaume François de l'Hospital (or l'Hôpital) (1661-1704).

Proposition4: (Hospital rule) (See [21]) $If\ f\ and\ g$ are two derivable functions in an interval$]a,b[$ except perhaps on $c\in]a,b[$ with :$f(c)=g(c)=0\ or\ \lim_{t\to c}f(t)=\infty\ and\ \lim_{t\to c}g(t)=\infty$, then: $\lim_{t\to c}\frac{f(t)}{g(t)}=\lim_{t\to c}\frac{f'(t)}{g'(t)}$

This can be extended to the cases: $t\to+\infty\ or\ t\to-\infty$ and: $t\to a^+\ or\ t\to b^-$

The process is repeated if the derivatives f' and g' satisfy the same conditions as f and g…

Proposition5: (See [9] p 32, [15], p 4) for (x_n) any sequence of $\mathbb{R}$, we have:

(1)$\mathrm{liminf}x_n=sup_{p\in\mathbb{N}}inf_{n\geq p}x_n$ exits always in $[-\infty,+\infty]$

(2)$a\leq x_n\leq b,\forall n\Rightarrow a\leq\mathrm{liminf}x_n\leq b$

(3)$a\leq\mathrm{liminf}x_n\Rightarrow\exists p\in\mathbb{N}\forall n\geq p: x_n\geq a$

(4) If (y_n) is a convergent sequence, then: $\mathrm{liminf}(x_n+y_n)=\mathrm{liminf}x_n+\lim_{n\to+\infty}y_n$

THE PROOF OF THE DE POLIGNAC CONJECTURE

Define for k an integer ≥ 1, n an integer ≥ 2 and m an integer ≥ 2, the following sets:

$I_{n,m}=\{(p,q)\in\mathbb{P}_{n+m}\times\mathbb{P}_n,\text{such that } 0\leq p-q\leq m\}$

$J_{n,2k}=\{(p,q)\in\mathbb{P}_{n+2k}\times\mathbb{P}_n,\text{such that } p-q=2k\}$

$J_{2k}=\{(p,q)\in\mathbb{P}\times\mathbb{P},\text{such that } p-q=2k\}$

The proof of the De Polignac conjecture will be deduced from the below lemmas.

Lemma1 : $\forall k\in\mathbb{N}^*\ \forall n$ integer ≥ 2, we have $J_{n,2k}\subset J_{2k}$

Proof : (of lemma1)

The result follows by construction of the involved sets.

Lemma2 : We have:

(1)$card(I_{n,m})=\sum_{q\in\mathbb{P}_n}(\pi(m+q)-\pi(q-1))$

(2)$card(J_{n,2k})=\sum_{q\in\mathbb{P}_n}(\pi(2k+q)-\pi(2k-1+q))$

Proof : (of lemma2)

(1)*We have : $I_{n,2k}=J_{n,2k}\cup I_{n,2k-1}$ with : $I_{n,2k-1}\cap J_{n,2k}=\emptyset$

*So : $card(J_{n,2k})=card(I_{n,2k})-card(I_{n,2k-1})$

*But :$I_{n,m}=\{(p,q)\in\mathbb{P}_{n+m}\times\mathbb{P}_n,\text{such that } q\leq p\leq q+m\}$

$=\cup_{q\in\mathbb{P}_n}((\mathbb{P}_{m+q}\backslash\mathbb{P}_{q-1})\times\{q\})$ (the sign: « \ » denotes the set difference)

*By the assertion (4) of proposition 1, we have :

$((\mathbb{P}_{m+s}\backslash\mathbb{P}_{s-1})\times\{s\})\cap((\mathbb{P}_{m+t}\backslash\mathbb{P}_{t-1})\times\{t\})=\emptyset$ for $s\neq t$ $\quad\Rightarrow card(I_{n,m})=\sum_{q\in\mathbb{P}_n}card(\mathbb{P}_{m+q}\backslash\mathbb{P}_{q-1})=\sum_{q\in\mathbb{P}_n}(\pi(m+q)-\pi(q-1))$

(2)Then :

$card(J_{n,2k})=\sum_{q\in\mathbb{P}_n}(\pi(2k+q)-\pi(q-1))-\sum_{q\in\mathbb{P}_n}(\pi(2k-1+q)-\pi(q-1))$

$=\sum_{q\in\mathbb{P}_n}(\pi(2k+q)-\pi(2k-1+q))$

Lemma3 : For $n\to+\infty$, we have :

$-(\frac{\pi(n)-\pi(2657)}{4\pi})\sqrt{2k+n}\ln(2k+n) \leq \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q)) \leq \frac{\pi(n)-\pi(2657)}{4\pi}\sqrt{2k+n}\ln(2k+n)$

Proof : (of lemma3)

*By the last result recalled in defintion2, we have :

$0 \leq \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q)) \leq \sum_{q=2}^{2657}(\pi(2k+q)-\pi(2k-1+q))$
$=\pi(2k+2657)-\pi(2k+1)$

*Because : $\lim_{n\to+\infty}(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n) = +\infty$, we have for n great :

$-\frac{(\pi(n)-\pi(2657))}{4\pi}\sqrt{2k+n}\ln(2k+n) \leq 0 \leq \pi(2k+2657)-\pi(2k+1) \leq \frac{(\pi(n)-\pi(2657))}{4\pi}\sqrt{2k+n}\ln(2k+n)$

(This is obtained by writing, for : $A = \pi(2k+2657)-\pi(2k+1) > 0$, the definition : $\lim_{n\to+\infty} x_n = +\infty \Leftrightarrow \forall A > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N\ x_n \geq A$, with $x_n = (\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)$. If: $\pi(2k+2657)-\pi(2k+1) = 0$, we have evidently: $\pi(2k+2657)-\pi(2k+1) = 0 \leq \frac{(\pi(n)-\pi(2657))}{4\pi}\sqrt{2k+n}\ln(2k+n)$)

*The result follows.

Lemma4 : For $n \to +\infty$ and any not null natural integer k, we have :

$card(J_{n,2k}) = \int_0^1(\sum_{q\in\mathbb{P}_n,q\geq2658}\frac{1}{\ln(t+2k+q-1)}) + O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))$

With :$\frac{|O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))|}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)} \leq \frac{1}{2\pi}$

Proof : (of lemma4)

*By lemma2, we have :

$card(J_{n,2k}) = \sum_{q\in\mathbb{P}_n}(\pi(2k+q)-\pi(2k-1+q))$
$=\sum_{q\in\mathbb{P}_n,q\geq2658}(\pi(2k+q)-\pi(2k-1+q)) + \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q))$

*But for: $q \geq 2658, \forall k \in \mathbb{N}^*$, we have: $2k+q \geq 2k-1+q \geq 2657$, so by the Schoenfeld inequality (Proposition3 of the ingredients):

$\int_0^{2k+q}\frac{dt}{\ln(t)} - \frac{1}{8\pi}(\sqrt{2k+q}\ln(2k+q)) \leq \pi(2k+q) \leq \int_0^{2k+q}\frac{dt}{\ln(t)} + \frac{1}{8\pi}(\sqrt{2k+q}\ln(2k+q))$ And

$\int_0^{2k-1+q}\frac{dt}{\ln(t)} - \frac{1}{8\pi}(\sqrt{2k-1+q}\ln(2k-1+q)) \leq \pi(2k-1+q) \leq \int_0^{2k-1+q}\frac{dt}{\ln(t)} + \frac{1}{8\pi}(\sqrt{2k-1+q}\ln(2k-1+q))$

*So, we have :

$\int_{2k-1+q}^{2k+q}\frac{dt}{\ln(t)} - \frac{\sqrt{2k+q}\ln(2k+q)}{4\pi} \leq \int_{2k-1+q}^{2k+q}\frac{dt}{\ln(t)} - (\frac{\sqrt{2k-1+q}\ln(2k-1+q)+\sqrt{2k+q}\ln(2k+q)}{8\pi}) \leq \pi(2k+q)-\pi(2k-1+q) \leq \int_{2k-1+q}^{2k+q}\frac{dt}{\ln(t)} + \frac{\sqrt{2k+q}\ln(2k+q)}{4\pi}$

*So, for a great n, we have:

$\sum_{q\in\mathbb{P}_n,q\geq2658}\int_{2k+q-1}^{2k+q}\frac{dt}{\ln(t)} + \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q)) - \frac{1}{4\pi}\sum_{q\in\mathbb{P}_n,q\geq2658}((\sqrt{2k+q}\ln(2k+q)) \leq card(J_{n,2k}) = \sum_{q\in\mathbb{P}_n}(\pi(2k+q)-\pi(2k-1+q)) = \sum_{q\in\mathbb{P}_n,q\geq2658}(\pi(2k+q)-\pi(2k-1+q)) + \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q)) \leq \sum_{q\in\mathbb{P}_n,q\geq2658}\int_{2k+q-1}^{2k+q}\frac{dt}{\ln(t)} + \sum_{q\in\mathbb{P}_{2657}}(\pi(2k+q)-\pi(2k-1+q)) + \frac{1}{4\pi}\sum_{q\in\mathbb{P}_n,q\geq2658}((\sqrt{2k+q}\ln(2k+q))$

*But:$\forall q \in \mathbb{P}_n$ (so: $q \leq n$), such that $q \geq 2658$ and $\forall k \in \mathbb{N}^*$, we have:

$\sqrt{2k+q}\ln(2k+q) \leq \sqrt{2k+n}\ln(2k+n)$

*So:$\sum_{q\in\mathbb{P}_n,q\geq2658}(\sqrt{2k+q}\ln(2k+q)) \leq card(\mathbb{P}_n\backslash\mathbb{P}_{2657})\sqrt{2k+n}\ln(2k+n)$
$=(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)$

*Then lemma3 gives, for n in the neighborhood of infinity and for any k a not nul natural integer :

$\sum_{q\in\mathbb{P}_n,q\geq2658}\int_{2k-1+q}^{2k+q}\frac{dt}{\ln(t)} - \frac{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}{2\pi} \leq card(J_{n,2k}) \leq \sum_{q\in\mathbb{P}_n,q\geq2658}\int_{2k-1+q}^{2k+q}\frac{dt}{\ln(t)} + \frac{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}{2\pi}$

*Finally, by the variable change: $t = s+2k+q-1$ in the involved integral, and according to definition 4, we have :

$card(J_{n,2k}) = \int_0^1\sum_{q\in\mathbb{P}_n,q\geq2658}\frac{ds}{\ln(s+2k-1+q)} + O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))$

Lemma5 : $\forall k \in \mathbb{N}^*\ \lim_{n\to+\infty}\int_0^1\sum_{q\in\mathbb{P}_n,q\geq2658}\frac{ds}{\ln(s+2k-1+q)} = +\infty$

ISSN 2349-0292
Impact Factor 3.802

Proof : (of lemma5)
* For $s \in [0,1]$, $q \in \mathbb{P}_n$ and $q \geq 2658$, we have : $\ln(s+2k-1+q) \leq \ln(2k+n)$
*So :$\frac{\pi(n)-\pi(2657)}{\ln(2k+n)} \leq \int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k-1+q)}$
($\sum_{q\in\mathbb{P}_n, q\geq 2658} 1 = card(\mathbb{P}_n \backslash \mathbb{P}_{2657}) = card(\mathbb{P}_n) - card(\mathbb{P}_{2657}) = \pi(n) - \pi(2657)$)
*But by the rarefaction law of prime numbers (proposition2 of the ingredients), and the Hôpital rule (proposition4 of the ingredients), we have:

$$\lim_{n\to+\infty} \frac{\pi(n)-\pi(2657)}{\ln(2k+n)} = \lim_{n\to+\infty} \frac{\pi(n)}{\ln(2k+n)} = \lim_{n\to+\infty} \frac{\pi(n)\ln(n)}{n} \frac{n}{\ln(n)\ln(n+2k)}$$
$$= \lim_{n\to+\infty} \frac{\pi(n)\ln(n)}{n} \lim_{n\to+\infty} \frac{n}{\ln(n)\ln(n+2k)} = \lim_{n\to+\infty} \frac{n}{\ln(n)\ln(2k+n)} = \lim_{n\to+\infty} \frac{1}{\frac{\ln(2k+n)}{n} + \frac{\ln(n)}{2k+n}} = \frac{1}{\lim_{n\to+\infty} \frac{\ln(n+2k)}{n} + \lim_{n\to+\infty} \frac{\ln(n)}{2k+n}} =$$
$$\frac{1}{\lim_{n\to+\infty} \frac{1}{n+2k} + \lim_{n\to+\infty} \frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{\frac{1}{2k+n} + \frac{1}{n}} = \lim_{n\to+\infty} \frac{n(2k+n)}{2n+2k} = \lim_{n\to+\infty} \frac{n^2}{2n} = \lim_{n\to+\infty} \frac{n}{2} = +\infty$$ for any not null k.

*So : $\lim_{n\to+\infty} \int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k+q-1)} = +\infty$

Lemma6 : We have :$\forall k \in \mathbb{N}^*$ $\lim_{n\to+\infty} \frac{\int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k+q-1)}}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)} = 0$

Proof : (of lemma6)
* For $s \in [0,1]$, $q \in \mathbb{P}_n$ and $q \geq 2658$, we have : $\ln(s+2k-1+q) \geq \ln(2k+2657)$
*So : $0 \leq \int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k-1+q)} \leq \frac{\pi(n)-\pi(2657)}{\ln(2k+2657)}$
*So : $0 \leq \frac{\int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k+q-1)}}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)} \leq \frac{1}{\ln(2k+2657)\sqrt{2k+n}\ln(2k+n)}$
*The result follows by letting $n \to +\infty$

Lemma7 : $\forall k \in \mathbb{N}^*$, We have :
$\mathrm{iminf}\left(\frac{\mathrm{card}(J_{n,2k})}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) = \mathrm{liminf}\left(\frac{O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right)$

Proof : (of lemma7)
The result is obtained by combination of the assertion (4) of proposition 5, lemma4 and lemma 6.

Lemma8 : We have :
(1)$0 \leq \mathrm{iminf}\left(\frac{\mathrm{card}(J_{n,2k})}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) = \mathrm{liminf}\left(\frac{O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) \leq \frac{1}{2\pi}$
(2)$\forall k \in \mathbb{N}^* \exists p_k \in \mathbb{N}$ Such that $\forall n \geq p_k$: $O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)) \geq 0$

Proof : (of lemma8)
(1)The result is obtained by combination of the assertion of proposition5, lemma7 and the last relation of lemma4.
(2)*By the assertion (2) of proposition 5, We have :
$\frac{\mathrm{card}(J_{n,2k})}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)} \geq 0$ $\forall k \in \mathbb{N}^*$ $\forall n \geq 2659 \Rightarrow \forall k \in \mathbb{N}^* \mathrm{liminf}\left(\frac{\mathrm{card}(J_{n,2k})}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) \geq 0$
*So, by lemma7, we have : $\forall \mathrm{k} \in \mathbb{N}^* \mathrm{liminf}\left(\frac{O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) \geq 0$
* The result follows, then, by use of the assertion 3 of proposition 5 and the fact that $(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n) > 0$ $\forall k \in \mathbb{N}^* \forall n \geq 2659$

Lemma9 : $\forall k \in \mathbb{N}^*$, we have : $\lim_{n\to+\infty} card(J_{n,2k}) = +\infty$

Proof : (of lemma9)
*By combination of lemma 4 and lemma8, we have :
$\forall k \in \mathbb{N}^* \exists p_k \in \mathbb{N}$ Such that : $\forall n \geq p_k$: $card(J_{n,2k}) \geq \int_0^1 \sum_{q\in\mathbb{P}_n, q\geq 2658} \frac{ds}{\ln(s+2k+q-1)}$
*Lemma5 gives the result by tending n to $+\infty$.

Theorem : (The De Polignac conjecture is true)For any not null natural integer k, it exists an infinite number of pairs of prime integers(p,q) such that : $p-q=2k$. I.e. $\forall k \in \mathbb{N}^*$ the set $J_{2k} = \{(p,q) \in \mathbb{P}\times\mathbb{P}, p-q=2k\}$ is infinite.

Proof : (of the theorem)
*Suppose contrarily that : $\exists k \in \mathbb{N}^*$ such that the set J_{2k} is finite (i.e.:$card(J_{2k}) < +\infty$).
*By lemma1, we have : $\forall n \geq 2$ $J_{n,2k} \subset J_{2k}$
*By the assertion (1) of proposition1, we have : $\forall n \geq 2$ $card(J_{n,2k}) \leq card(J_{2k})$
*But, then, by lemma5 : $\lim_{n\to+\infty} card(J_{n,2k}) = +\infty \leq card(J_{2k}) < +\infty$
*This being impossible the theorem is proved.

ISSN 2349-0292
Impact Factor 3.802

DEDUCTION OF THE TWIN PRIMES CONJECTURE

Corollary :(the twin primes conjecture is true) It exists an infinite number of prime integers p such that $p + 2$ is also prime.

Proof : (of the Corollary)

[1] The result is obtained by letting $k = 1$ in the above theorem.

REFERENCES

[1] Caldwell, C.: liste des nombres premiers jumeaux. Available at: http://primes.utm.edu/top20.php?id=1 (Accessed on: July 4, 2018)
[2] Chen, Jingrun (1966): on the representation of a large even integer as the sum of a prime and the product of at most two primes (Chinese). J. Kexue Tongbao, 385-386
[3] De La vallée Poussin, Ch.J.G.N (1896) : Recherches analytiques, la théorie des nombres premiers. Ann. Soc. Scient. Bruxelles 20, pp185-256.
[4] De Polignac, A. (1849) : Six propositions arithmétiques déduites du crible d'Eratosthène. Nouv. Ann. Math 8, pp423-429
[5] De Polignac, A. (1849): Comptes rendus Paris 29, 400, pp738-739
[6] Erdos, P. (1973): The Art of counting selected writings. Edited by Joel Spencer. Cambridge, Mass, MIT press.
[7] Euclide(1966):Les éléments. Traduction Peyrard. Blanchard. Paris.
[8] Friedlander, John (2015): prime numbers: a much needed gap is finally found. Notices. Amer. Math .Soc, Vol 62, n°6, June 2015.
[9] Flory, G. (1978): Topologie, Analyse: Exercises avec solutions, Tome I. Vuibert. Paris.
[10] Ghanim, Mohammed (2017): Confirmation of the Riemann hypothesis. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), in the" April 2017" issue, India. Available at: www.gjaets.org/ (Accessed on: July 3, 2018)
[11] Ghanim, Mohammed (2018): Confirmation of the Goldbach binary conjecture. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), in the « May 2018 » issue, India.Available at : www.gjaets.org/ (Accessed on : July 3, 2018)
[12] Hadamard, J.S. (1896) : Sur la distribution des zéros de la fonction ζ(s) est ses conséquences arithmétiques. Bull.Soc.Math.France24, pp199-220
[13] Hardy and Littlewood (1922): Some problems of partitio numerum III: on the expression of a number as a sum of primes. Acta Mathematica Vol 44, pp1-70.
[14] Hilbert, David (1900): Mathematical problems. Lecture delivered before the international congress of mathematicians at Paris.Available at : www2.clarku.edu/~djoyce/Hilbert/(Accessed on: July 4, 2018)
[15] Jean, Roger (1982): Mesure et integration. Presses de l'université du Québec. Canada.
[16] Nouvelles: www.futura.sciences.com/news-nombres-premiers(Accessed on: July 4, 2018)
[17] Planet math: Cardinality of disjoint union of finite sets. Available at: Planetmath.org/Cardinalityofdisjointunionoffinitesets/ (Accessed on: July 3, 2018)
[18] Yitang, Zhang (2014): "Gaps between primes". Annals of maths, Vol 179, issue 3, pp 1121-1174. Available: annals.math.princeton.edu/2014/179-3/p07 (Accessed on: July 4, 2018)
[19] Weisstein, Eric. W. :Twin primes. From Mathworld- A Wolfram web resource.Available: http://mathworld.wolfram.com/TwinPrimes.html(Accessed on: July 4, 2018)
[20] Wikipedia: Cardinality. Available at: https://en.wikipedia.org/Cardinality/ (Accessed on: July 3, 2018)
[21] Wikipedia: L'Hôpital rule. Available at: https://en.wikipedia.org/wiki/L%27Hopital%27s_rule/ (Accessed on: July 3, 2018)
[22] Wikipedia: Prime counting function.Available at: https://en.wikipedia.org/wiki/Prime-counting_function/ (Accessed on: July 3, 2018)
[23] Wikipedia: Asymptotic analysis. Available at: https://en.wkipedia.org/wiki/Asymptotic_analysis/ (Accessed on July 4, 2018)