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A VARIANT OF THE RIEMANN HYPOTHESIS'S PROOF USING THE ROBIN CRITERION WITH DEDUCTION OF SOME CONSEQUENCES

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ABSTRACT

I have published, in the GJAETS's April 2017 issue [22], a proof of the Riemann Hypothesis, remained open since 1859 and saying that "if ζ is the zeta function, defined on $A = \{\ z = x + iy \in \mathbb{C}\ \ (i^2 = -1)$, such that: $Re(z) = x \neq 1\}$, then: $\zeta(z) = \zeta(x + iy) = 0,\ z \neq -2k, k \in \mathbb{N}^* \Rightarrow Re\ (z) = x = \frac{1}{2}$" using the Schoenfeld criterion saying that "if $\pi(x)$ is the prime counting function, then: $\forall x \geq 2657\ \ |\pi(x) - \int_0^x \frac{dt}{\ln(t)}| \leq \frac{\sqrt{x}\ln(x)}{8\pi}$ if and only if the Riemann Hypothesis is true". Now I give a variant of the proof of the Riemann Hypothesis using the Robin criterion saying: "the Riemann Hypothesis is true if and only if $\forall n$ integer ≥ 5041: $\sum_{d|n} d \leq ne^{\gamma}\ln(\ln(n))$, where the sum runs all the divisors d of n and $\gamma = 0.5772$ .... is the Euler-Mascheroni constant". The proof is elementary based on the connectivity of intervals and the fact that the diameter of a non empty boundary of a set is equal to the diameter of this set in a normed space. The proof uses some techniques of my previous papers [28], [29].

KEYWORDS: prime integer, prime-counting function, Riemann Hypothesis, Robin criterion, Schoenfeld criterion, maximum-minimum, integer part function, continuity, diameter of a set, the boundary of a set, the adherence of a set, the interior of a set 2010 Mathematics Subject Classification: A 11 xx (Number theory)

INTRODUCTION

<u>Definition 1:</u> We call "zeta function" the complex number ζ, defined for any complex number $z = x + iy$ with $i^2 = -1$, such that $Re(z) = x \neq 1$ by:

$$\zeta(z) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^z} \text{ for: } Re(z) > 1 \\ \dfrac{\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^z}}{1 - 2^{1-z}} \text{ for: } 0 < Re(z) < 1 \\ 0 \text{ for: } z = -2k, k \in \mathbb{N}^* \\ \dfrac{\sum_{n=1}^{+\infty} \frac{1}{n^{1-z}}}{2\cos\left(\frac{\pi}{2}z\right)\int_0^{+\infty} t^{z-1}e^{-t}dt} \text{ for: } Re(z) < 0 \text{ and } z \notin -2\mathbb{N}^* \\ -\frac{1}{2} \text{ for: } z = 0 \\ 1 \text{ is a simple pole such that: } \lim_{z\to 1}(z-1)\zeta(z) = 0 \end{cases}$$

<u>Definition 2:</u> if ζ is the zeta function given by definition 1, we call Riemann hypothesis (RH) the following assertion:

$$\boldsymbol{\zeta(z) = \zeta(x + iy) = 0, Re(z) \neq 1, z \neq -2k, k \in \mathbb{N}^* \Rightarrow Re\ (z) = x = \frac{1}{2}}$$

<u>Definition3:</u> We call Robin criterion the following assertion:"the two below propositions are equivalent:
(1)The Riemann Hypothesis.

(2) $\forall n$ integer ≥ 5041: $\sigma(n) = \sum_{d|n} d \leq ne^{\gamma}\ln(\ln(n))$, with $\gamma = 0.7552$ ...is the Euler-Mascheroni constant proved to be irrational by Mohammed Ghanim in [23] and the sum runs all the divisors d of the integer n".

<u>History:</u> In 1859, the German mathematician Bernhard Riemann (1826-1866) conjectured his hypothesis. He wrote in [47]:«… es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien. ».
In 1913, the Swedish mathematician Thomas Hakon Gronwall (1877-1932) proved, in [30], that

$\limsup_{n\to+\infty} \frac{\sigma(n)}{n\ln(\ln(n))} = e^{\gamma}$

In 1915, the Indian mathematician Srinivasa Ramanujan (1887-1920) proved, in [45], that:

$$\text{The Riemann Hypothesis true} \Rightarrow \forall n \geq 5041 \ \frac{\sigma(n)}{n\ln(\ln(n))} \leq e^{\gamma}$$

In 1984, the French mathematician Guy Robin made the striking discovery (See [48]) that a reverse implication is true:

$$\forall\, n > 7!\ \sigma(n) < ne^{\gamma}\ln(\ln(n)) \Rightarrow \text{RH is true}$$

In 2002, The American mathematician Jeffrey Clark Lagarias (1949-) eliminated the Euler-Mascheroni constant γ in the Robin criterion and proved that the two following assertions are equivalent (See [36]):
(1)The Riemann Hypothesis is true

(2)$\forall n \geq 1\ \sigma(n) = \sum_{d|n} d \leq \sum_{k=1}^{n}\frac{1}{k} + e^{\sum_{k=1}^{n}\frac{1}{k}}\ln\left(\sum_{k=1}^{n}\frac{1}{k}\right)$

The note: my purpose in the present brief note is to show the Riemann Hypothesis by using the Robin inequality. The proof is elementary using the elementary topological properties satisfied by the integer part function, the connectivity of intervals and the fact that the diameter of a non empty boundary of a set is equal to the diameter of this set in a normed space. The proof uses some techniques of my previous papers [28], [29].

The main result of the paper is:

Theorem: for any integer n such that $p_n \geq 5041 = 71^2$ we have:
$\sigma(n) = \sum_{d|n} d \leq ne^{\gamma}\ln(\ln(n))$, with $\gamma = 0.7552$ ...is the Euler-Mascheroni constant showed to be irrational by M.Ghanim in [23].

Methods: *I show, first, that the Robin inequality is true for $n \in \mathbb{N} \cap [5041, 5050]$
*Then I consider the case: $n \in \mathbb{N} \cap [5051, +\infty[$
*Noting that:$n \in [5051, +\infty[= \bigcup_{m=676}^{+\infty}[p_m, p_{m+1}[$, where $(p_n)_{n\geq 1}$ is the strictly increasing sequence of prime integers, we have: $\exists m \geq 676$ such that: $n \in [p_m, p_{m+1}[$, I consider the sets:
$A_m = \{t \in [p_m, p_{m+1}],\ \sum_{E\left(\frac{-E(-t)}{d}+\frac{1}{2}\right)=\frac{-E(-t)}{d}} d \leq -E(-t)e^{\gamma}\ln(\ln(-E(-t)))\}$
*using connectivity of the real interval $[p_m, p_{m+1}]$, I show that:$A_m = [p_m, p_{m+1}]$,.
*So: $\forall n$ integer ≥ 5041 $\sigma(n) = \sum_{d|n} d \leq ne^{\gamma}\ln(\ln(n))$,

Organization of The paper: The paper is organized as follows. The §1 is an introduction containing the necessary definitions and some History. The §2 contains the ingredients of the proofs. The §3 contains the proof of our main result. The §4 contains some consequences. The §5 contains the references of the paper given for further reading.

INGREDIENTS OF THE PROOFS

Notation: the closed, the semi-open and the open intervals of $\mathbb{R}$, are (respectively) denoted as below (if $a < b$):
$[a,b]$=$\{t \in \mathbb{R}, a \leq t \leq b\}$,$[a,b[$=$\{t \in \mathbb{R}, a \leq t < b\}$,},$]a,b]$=$\{t \in \mathbb{R}, a < t \leq b\}$, $]a,b[$=$\{t \in \mathbb{R}, a < t < b\}$
Remark that: [a, a] = {a}

Definition4: (The absolute value [71]) the absolute value function ||is defined on $\mathbb{R}$ by: $|t| = \begin{cases} t\ if\ t > 0 \\ 0\ if\ t = 0 \\ -t\ if\ t < 0 \end{cases}$.

We have: $|x| \leq a \Leftrightarrow -a \leq x \leq a$
Definition 5: (division and divisors [64]) let $b, c \in \mathbb{N}^*$, b divides c (denoted by: $b|c$) $\Leftrightarrow \exists a \in \mathbb{N}^*$ such that: $c = ab$. We denote, for $c \in \mathbb{N}^*$, the set of divisors of c by $D(c)$.

Definition 6: (prime integers [63]) a positive integer p is called to be prime if its set of divisors is $D(p) = \{1, p\}$. The set of all prime integers is denoted $\mathbb{P}$. For $t \geq 2$, we define the set: $\mathbb{P}_t = \{p \in \mathbb{P}, p \leq t\}$ which is a finite set having the cardinal (the number of elements): $card(\mathbb{P}_t) = \pi(t)$ called the prime-counting function.

Proposition1: (Euclid [21]) the set $\mathbb{P}$ of prime integers is a strictly increasing infinite sequence $(p_n)_{n\geq 1} = (2,3,5,7,11,13,17,\dots)$.

Proposition 2: (See [40]): $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, p_{10} = 29, p_{13} = 41,\ p_{25} = 97, p_{26} = 101, p_{27} = 103, p_{115} = 631, p_{169} = 1009, p_{368} = 2521, p_{676} = 5051$.

Proposition3: (See [12]) we have the following decompositions of the integers $5041 \leq n \leq 5050$ in prime factors:
(1)5041=71^2 (2)5042=2× 2521 (3)5043=3× 41^2 (4)5044=$2^2 \times 13 \times 97$ (5)5045=5× 1009 (6)5046=2× 3 × 29^2 (7)5047=$7^2 \times 103$ (8)$5048 = 2^3 \times 631$ (9)5049=$3^3 \times 11 \times 17$ (10)$5050 = 2 \times 5^2 \times 101$.

Proposition 4: We give below the sets of divisors $D(n)$ and the sums: $\sum_{d|n} d$ for: $5041 \leq n \leq 5050$:
(1)$D(5041) = \{1,71,5041\} \Rightarrow \sum_{d|5041} d = \sum_{d\in D(5041)} d = 1 + 71 + 5041 = 5113$
(2)$D(5042) = \{1,2,2521,5042\} \Rightarrow \sum_{d|5042} d = \sum_{d\in D(5042)} d = 1 + 2 + 2521 + 5042 = 7566$
(3)$D(5043) = \{1,3,41,123,1681,5043\} \Rightarrow \sum_{d|5043} d = \sum_{d\in D(5043)} d = 1 + 3 + 41 + 123 + 1681 + 5043 = 6892$
(4)$D(5044) = \{1, 2, 4, 13, 26, 52, 97, 194, 388, 1261, 2522, 5044\}$
$\Rightarrow \sum_{d|5044} d = \sum_{d\in D(5044)} d = 1 + 2 + 4 + 13 + 26 + 52 + 97 + 194 + 388 + 1261 + 2522 + 5044 = 8343$
(5)$D(5045) = \{1,5,1009,5045\} \Rightarrow \sum_{d|5045} d = \sum_{d\in D(5045)} d = 1 + 5 + 1009 + 5045 = 6060$
(6)$D(5046) = \{1,2,3,6,29,58,87,174,841,1682,2523,5046\}$
$\Rightarrow \sum_{d|5046} d = \sum_{d\in D(5046)} d = 1 + 2 + 3 + 6 + 29 + 58 + 87 + 174 + 841 + 1682 + 2523 + 5046 = 9611$
(7)$D(5047) = \{1,7,49, 103, 721, 5047\} \Rightarrow \sum_{d|5047} d = \sum_{d\in D(5047)} d = 1 + 7 + 49 + 103 + 721 + 5047 = 5928$
(8)$D(5048) = \{1,2,4,8,631,1262,1484,5048\}$
$\Rightarrow \sum_{d|5048} d = \sum_{d\in D(5048)} d = 1 + 2 + 4 + 8 + 631 + 1262 + 1484 + 5048 = 7178$
(9)$D(5049) = \{1,3,9,11,17,27,33,51,99,153,187,297, 459,561,1683,5049\}$
$\Rightarrow \sum_{d|5049} d = \sum_{d\in D(5049)} d = 1 + 3 + 9 + 11 + 17 + 27 + 33 + 51 + 99 + 153 + 187 + 297 + 459 + 561 + 1683 + 5049$ =8640
(10)$D(5050) = \{1,2,5,10,11,22,25,50,55,110,275,5050\}$
$\Rightarrow \sum_{d|5050} d = \sum_{d\in D(5050)} d = 1 + 2 + 5 + 10 + 11 + 22 + 25 + 50 + 55 + 110 + 275 + 5050 = 6111$

Proposition5: ([21], [40]) we have:$[5051, +\infty[= \cup_{m=676}^{+\infty} [p_m, p_{m+1}[$

Définition7: ([57], [65]) we note, for$x \in \mathbb{R}$, by $E(x) \in \mathbb{Z}$ the integer part of the real x i.e. the single integer $E(x)$such that: $E(x) \leq x < E(x) + 1$

Proposition6: ([57], [65]) we have:
(i)$\forall x \in \mathbb{Z}: E(x) = x$ and $E(-x) = -x$
(ii)$\forall x \in \mathbb{R} - \mathbb{Z}\ \ E(-x) = -E(x) - 1$
(ii) $0 \leq x < 1 \Rightarrow E(x) = 0$
(iii)$\forall x \in \mathbb{R}: 0 \leq x - E(x) < 1$
(iv) $\forall x, y \in \mathbb{R}\ E(x + y) = E(x) + E(y) + \chi_{[1,2[}(x - E(x) + y - E(y))$
Where: $\chi_{[1,2[}(t) = \begin{cases} 1\ if\ t \in [1,2[\\ 0\ if\ t \notin [1,2[\end{cases}$ is the characteristic function of the interval [1, 2[
In particular:$\forall x \in \mathbb{Z} \forall y \in \mathbb{R}\ E(x + y) = x + E(y)$

Example: $E\left(x + \frac{1}{2}\right) = E(x) + \chi_{[1,2[}\left(\frac{1}{2} + x - E(x)\right) = E(x) or\ E(x) + 1$
(v)$\forall x, y \in \mathbb{R}\ \ x < y \Rightarrow E(x) \leq E(y)$

Definition8: If A is any subset of any set X, we define the subset $A^c = \{x \in X, x \notin A\}$ called the complementary set of A.

Definition 9: (definition of a topologic space [58]) (1) A topologic space X is a set equipped with a part :$\tau \subset P(X)$ (the set of its parts) called topology such that:
(i) $X, \emptyset \in \tau$
(ii) Any arbitrary (finite or infinite) union of members of τ (i.e. open subsets) still belongs to τ (i.e. is open)
(iii) The intersection of any finite number of members of τ (i.e. open subsets) still belongs to τ (i.e. is open)
(2)An element U of τ is called an open subset of X
(3)For $U \in \tau$: U^c is called a closed subset of X

Proposition 8: ([58]) we have:
(i) Any arbitrary (finite or infinite) intersection of closed subsets of X is still closed.
(ii) The union of any finite number of closed subsets is still closed.

Definition10: (definition of the interior [66]) we call interior of a subset A of a topological space X, denoted int(A), the set:

$$int(A) = \bigcup_{\text{all the open subsets O of X} \subset A} O$$

Int (A) is the greatest open subset contained in A.
Int $(A) = \{\boldsymbol{x} \in X, \exists O(x)$ an open subset of X such that $x \in O(x) \subset A\}$.
For $(\mathbb{R}, ||)$: int $(A) = \{x \in \mathbb{R}, \exists h > 0$such that $[x - h, x + h] \subset A\}$.
If Y is a subspace of X (equipped with the induced topology) and $A \subset Y$, then:
The interior of A relatively to Y is $=Y \cap int(A)$

Proposition9: ([66]) (i) $A \supset int(A)$(ii) $int(X) = X, int(\emptyset) = \emptyset$(iii)$int(\bigcap_{k=1}^{m} A_k) = \bigcap_{k=1}^{m} int(A_k)$
(iv) For any finite subset $A \subset (\mathbb{R}, ||)$, we have: $int(A) = \emptyset$
(iv) $A \subset B \Rightarrow int(A) \subset int(B)(v)A$ open$\Leftrightarrow int(A) = A$

Definition11: ([59]) we call adherence of a subset A of a topological space X, denoted adh (A), the set:

$$adh(A) = \bigcap_{all\ the\ closed\ subsets\ F of\ X \supset A} F$$

adh (A) is the smallest closed subset containing A.
If X is a metrical space $adh(A) = \left\{a \in X, \exists a_n \in A: a = \lim_{n \to +\infty} a_n\right\}$.
If Y is a subspace of X (equipped with the induced topology) and $A \subset Y$, then:
The adherence of A relatively to Y is $=Y \cap adh(A)$

Example: if $[a, b] \subset [c, d]$, the adherence of [a, b [relatively to [c, d] is = [a, b]

Proposition 10: ([59]) (i) $A \subset \text{adh}(A)$(ii) $\text{adh}(X) = X, \text{adh}(\emptyset) = \emptyset$(iii)$\text{adh}(\bigcup_{k=1}^{m} A_k) = \bigcup_{k=1}^{m} \text{adh}(A_k)$
(iv) $A \subset B \Rightarrow \text{adh}(A) \subset \text{adh}(B)$ (v) A closed$\Leftrightarrow \text{adh}(A) = A$
(vi) $(\text{adh}(A))^c = \text{int}(A^c)$ (vii) $(\text{int}(A))^c = \text{adh}(A^c)$

Definition 12: (A metric space [73]) A **metric space** is an ordered pair (E, d) where E is a set and d is a metric on E, i.e., a function : $d: E \times E \to \mathbb{R}^+$ such that : $\forall (x, y) \in E \times E$
(1)$d(x, y) = 0 \Leftrightarrow x = y$
(2)$d(x, y) = d(y, x)$
(3)$d(x, z) \leq d(x, y) + d(y, z)$

Proposition 11: ([73]) A metric defines a topology on a metric space (E, d) as follows:
U open $\Leftrightarrow \forall x \in U \exists \epsilon(x)$such that: $B(x, \epsilon(x)) = \{t \in E, d(x, t) < \epsilon(x)\} \subset U$
In particular: on $(\mathbb{R}, ||)$: U open $\Leftrightarrow \forall x \in U \exists \epsilon(x) > 0$ such that: $]x - \epsilon(x), x + \epsilon(x)[\subset U$

Proposition 12 : (the completion of metric space) : for any metric space $(E, d) \exists F$ a metric space such that :

$$E \subset F$$
$$\text{adh}\ (E) = F$$

Definition13: (continuity [60]) a function $f: X \to Y$ between two metrical spaces X, Y is continuous in t $\in X \Leftrightarrow$

$(\forall (t_n)_n \subset X: (\lim_{n\to+\infty} t_n = t) \Rightarrow (\lim_{n\to+\infty} f(t_n) = f(t)))$

Proposition 13: ([57], [65]) (1) the function integer part E is continuous on the set: $\mathbb{R} - \mathbb{Z}$ (the complementary in $\mathbb{R}$ of $\mathbb{Z}$).

(2)E is not continuous in: $n \in \mathbb{Z}$, because we have: $\begin{cases} \lim_{h\to 0^+} E(n+h) = n \\ \lim_{h\to 0^-} E(n+h) = n-1 \end{cases}$

Proposition14: (negation of a proposition [55], [62]) the negation of a proposition (P), denoted non (P), is the proposition true when (P) is false and false when (P) is true. We have: non (non (P)) = (P).

Example: non $(\forall) = \exists$, non $(\exists) = \forall$, non (=) =$\neq$, non (<) =$\geq$, non (or) =and

Definition 14 : (definition of a group [74]) A group is a set G together with a binary operation on G, here denoted ".", that combines any two elements a and b to form an element of G, denoted a.b, such that the following three requirements, known as *group axioms*, are satisfied:
1) Associativity :$\forall a, b, c \in G \ \ (a.b).c = a.(b.c)$
2) Identity element :$\exists e \in G$ such that $\forall a \in G \ a.e = e.a = a$
A such element is unique, it is called the identity element of the group
3) Inverse element :$\forall a \in G \exists b = a^{-1} \in G$ such that $a.b = b.a = e$ with e the identity element.
The element a^{-1} associated to a is unique and called the inverse element of the element a.

Definition 15 : (definition of a ring [75]) A **ring** is a set R equipped with two binary operations + (addition) and · (multiplication) satisfying the following three sets of axioms, called the **ring axioms :**
*R is an Abelian group under addition, meaning that:
(i) $(a + b) + c = a + (b + c)$ for all a, b, c in R (that is, + is associative).
(ii) $a + b = b + a$ for all a, b in R (that is, + is commutative).
(iii)There is an element 0 in R such that $a + 0 = a$ for all a in R (that is, 0 is the additive identity).
(iv)For each a in R there exists $-a$ in R such that $a + (-a) = 0$ (that is, $-a$ is the additive inverse of a).
*R is a monoid under multiplication, meaning that:
(v) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in R (that is, · is associative).
(vi)There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ for all a in R (that is, 1 is the multiplicative identity).
*Multiplication is distributive with respect to addition, meaning that:
(vii)$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, c in R (left distributivity).
(viii)$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in R (right distributivity).

Example: the ring $\mathbb{Z}/n\mathbb{Z}$ = {the class of elements having all the same rest for the Euclidean division by n}.

Definition 16: (definition of a field [67]) a field is a set F together with two binary operations on F called *addition* and *multiplication*. A binary operation on F is a mapping $F \times F \to F$, that is, a correspondence that associates with each ordered pair of elements of F a uniquely determined element of F. The result of the addition of a and b is called the sum of a and b, and is denoted $a + b$. Similarly, the result of the multiplication of a and b is called the product of a and b, and is denoted ab or $a \cdot b$. These operations are required to satisfy the following properties, referred to as *field axioms* (in these axioms, a, b, and c are arbitrary elements of the field F):
Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Commutativity of addition and multiplication: $a + b = b + a$, and $a \cdot b = b \cdot a$.
Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that:

$$a + 0 = a \text{ and } a \cdot 1 = a.$$

Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called the *additive inverse* of a, such that $a + (-a) = 0$.
Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1} or $1/a$, called the *multiplicative inverse* of a, such that $a \cdot a^{-1} = 1$.
Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Definition 17: (ultra-metric absolute value [76]) an ultra-metric absolute value is an application: $\|: K \to \mathbb{R}^+$,

with K is a field, such that:
1.$\forall x \in K \quad |x| = 0_{\mathbb{R}} \Leftrightarrow x = 0_K$
2.$\forall x, y \in K \quad |xy| = |x||y|$
3. $\forall x, y \in K\ |x + y| \leq \max(|x|, |y|)$

Definition 18: (definition of a vector space [68]) a Vector space E (or a linear space) on field F (set of scalars) is a set of objects called vectors which may be added with the addition operation (+) and multiplied ("scaled") with the multiplication operation (.) by numbers such that:

u + (**v** + **w**) = (**u** + **v**) + **w 2) u** + **v** = **v** + **u**
2) There exists an element **0** ∈ *V*, called the *zero vector*, such that **v** + **0** = **v** for all **v** ∈ *V*.
3) For every **v** ∈ *V*, there exists an element −**v** ∈ *V*, called the *additive inverse* of **v**, such that **v** + (−**v**) = **0**.
4) 1**v** = **v**, where 1 denotes the multiplicative identity in *F*.
5) *a*(**u** + **v**) = *a***u** + *a***v**
6) (*a* + *b*) **v** = *a***v** + *b***v**
7) *a*(*b***v**) = (*ab*)**v**

Definition 19: (definition of a normed space [69]) a normed vector space is a vector space E on the field F=$\mathbb{R}$ *or* $\mathbb{C}$ equipped with a norm ||.||such that:
(1)$\forall x \in E\ \ ||x|| \geq 0$
(2) $||x|| = 0 \Rightarrow x = 0$
(3)$\forall \alpha \in F\ ||\alpha x|| = |\alpha|||x||$
(4)$\forall x, y \in E\ ||x + y|| \leq ||x|| + ||y||$

Definition 20: (definition of a Hilbert space [77]) H is a space Hilbert space if it is a vector space on $\mathbb{R}$ or $\mathbb{C}$ on which there is an inner product $<< x, y >$ associating a complex number to each pair of elements $(x, y) \in H^2$ such that:
1.$\forall x, y \in H\ \ < y, x >= \overline{< x, y >}$ where $z = a + ib \in \mathbb{C} \Rightarrow \bar{z} = a - ib$
2.$\forall x \in H < x, x > \in R$
3.$\forall x \in H$: $x \neq 0 \Rightarrow < x, x > > 0$ and $x = 0 \Rightarrow < x, x >= 0$
4.$\forall a, b \in \mathbb{C} \forall x_1, x_2, y \in H\ \ < ax_1 + bx_2, y \geq a < x_1, y > +b < x_2, y >$
5.$\forall a, b \in \mathbb{C} \forall y_1, y_2, x \in H\ \ < x, ay_1 + by_2 \geq \bar{a} < x, y_1 > +\bar{b} < x, y_1 >$
6. (The Cauchy-Schwartz inequality) : $\forall x, y \in H < x, y > \leq ||x||||y||$
7. The inner product defines on H a norm and a distance making it a normed complete space i.e. :
(i)$||x|| = \sqrt{< x, x >}$ and $d(x, y) = ||x - y|| = \sqrt{< x - y, x - y >}$
(ii)$\forall (u_k)_{k \geq 0} \subset H$: $\sum_{k=0}^{+\infty} ||u_k|| < +\infty \Rightarrow \sum_{k=0}^{+\infty} u_k$ converges in H.

Definition21: (definition of the boundary of a subset [70]) in a topological space X we call the boundary of a subset A, the subset:

$$\mathrm{Fr}(A) = \mathrm{adh}\,(A) - \mathrm{int}(A) = \mathrm{adh}(A) \cap (\mathrm{int}(A))^c$$

Fr (A) ={x∈ X, $\forall O(x)$ a neighborhood of x, we have: $O \cap A \neq \emptyset$ *and* $O \cap A^c \neq \emptyset$}
For a normed vectorial space if $B(x, \epsilon) = \{t \in X, ||x - t|| \leq \epsilon\}$, we have:
Fr (A) = {x∈ X, $\forall \epsilon > 0$ we have: $B(x, \epsilon) \cap A \neq \emptyset$ *and* $B(x, \epsilon) \cap A^c \neq \emptyset$}

Proposition 15: We have: adh (A) = int(A) ∪ Fr(A)

Proposition 16: (if Fr (A)≠ ∅: diam (A)=diam (Fr(A)) [7]) Let (E, ||. ||) a normed vector space and A a non empty bounded subset of E. Define :
diam (A)=sup {‖y − x‖, x, y ∈A}
Then:
1. adh (A) and Fr (A) are also bounded.
2. a) We have: diam (A) =diam (adh(A)).
b) we have diam (int(A))) ⊂diam(A), the inclusion coulding be strict.
3. a) We have: diam (Fr(A)) ≤diam(A).
b) Let x ∈ A, and u ∈ E with u≠0. The set X =X(x, u) = {t≥0|x+tu∈A} has a supremum sup(X).
c) If Fr (A)≠ ∅, then any half -line L(x, u) issued from a point x ∈ A and having u≠ 0 for directory coefficient is

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such that Fr (A)∩ $L(x, u) \neq \emptyset$.
d) If Fr(A) $\neq \emptyset$, we have: diam (Fr (A)) =diam (A).

Proof: (of proposition 16)
1. Let M such that A⊂ B (0, M) = $\{x\in E, ||x|| \leq M\}$. Let x ∈ adh (A) and $(x_n)_n$ a sequence of A converging to x. Then:
*we have: $\forall n \in \mathbb{N}$ $\|x_n\|\leq M \Longrightarrow \lim_{n\to+\infty} ||x_n|| = \|x\|\leq M$.
* So: adh (A) is bounded.
* Fr (A) ⊂adh (A)⇒ Fr (A) is also bounded.
2. we have : int(A) ⊂A⊂ adh(A)⇒ diam (int(A)) ≤diam(A) ≤diam (adh(A)).
a)*We have: diam (A)=diam (adh(A)).
*Indeed:
** By definition of the supremum $\forall\ \epsilon > 0 \exists x, y\ \in adh(A)$ such that: ‖x−y‖≥diam (adh (A)) $-\ \epsilon$
**By definition of adh (A) : ∃a, b ∈ A such that : ‖x− a‖≤ϵ et ‖y− b‖≤ϵ
**By the triangular inequality, we have: ‖x− y‖=‖x− a+ a− b+ b− y‖≤‖ x − a||+||a′ − b|| + || b − y ‖
**So: a, b ∈ A and x, y∈adh (A)⇒ diam (A)≥‖a− b‖≥‖x− y‖−‖a− x‖−‖b− y‖
≥ diam (adh (A)) − $\epsilon - \epsilon - \epsilon$ =diam (adh (A)) − 3 ϵ
** So : $\forall\epsilon > 0$ we have: diam (adh (A)) ≥diam (A) ≥ diam (adh (A)) − 3 ϵ.
**Finally, tending $\epsilon \to 0$, we get well: diam (A) =diam (adh (A)).
b) For: E=ℝ and A= [0, 1] ∪ {2}, we have:
int (A)=] 0,1 [and : diam (int(A))=1, diam(A)=2, so : diam (int(A))≠ diam(A)
3.a)by 2)a) we have : Fr(A) ⊂ adh(A)⟹diam(Fr(A))≤diam(adh(A))=diam(A)
b) *A being bounded:$\exists M > 0$ such that A⊂B (0, M).
*X is a non empty subset of ℝ bounded above by: $\frac{M+\|x\|}{\|u\|}$ (u≠ 0)
*Indeed: we have ∀t∈X:‖x+ tu‖≤M⟹‖tu‖−‖x‖≤M⟹t‖u‖≤M+‖x‖⇒ $0 \leq t \leq \frac{M+\|x\|}{\|u\|}$,
*So, X has a supremum sup (X) =m.
c)* by definition of a half-line issued from x having $u \neq 0$ for directory coefficient, we have L(x, u) = {x+tu; t≥0}.
*Suppose Fr (A)≠ ∅
*if sup (X) =m, show that: x+mu ∈Fr (A).
*Indeed, ∀ϵ>0:
**by definition of the supremum written for$\frac{\epsilon}{||u||}$:∃ t_1 ∈ X(x, u) =X such that: m $-\frac{\epsilon}{||u||}$ <t_1<m
**By definition of t_1, we have: $t_1 \geq 0$ and $x + ut_1$ ∈A
**We have: $\|x+mu-(x+t_1u)\| = (m - t_1)\|u\| \leq \frac{\epsilon}{||u||}\|u\| = \epsilon$
**So: $x + t_1u$ ∈B(x+mu, ϵ) $\cap A$.
**By definition of m as a supremum of X=X(x, u) we have:
$$m < m + \frac{\epsilon}{||u||} \Rightarrow m + \frac{\epsilon}{||u||} \notin X(x,u) \Rightarrow x + (m+\frac{\epsilon}{||u||})\, u \notin A \Rightarrow x + (m + \frac{\epsilon}{||u||})u \in A^c$$
**We have: $\|x+mu-(x + (m+\frac{\epsilon}{||u||}) u)\| = \frac{\epsilon}{||u||}\|u\| = \epsilon \Rightarrow (x + (m + \frac{\epsilon}{||u||})u) \in A^c \cap B(x + mu, \epsilon)$
**So, by definition 12: $\forall\epsilon > 0$ B(x+mu,ϵ) $\cap A \neq \emptyset$ and $A^c \cap B(x + mu, \epsilon) \neq \emptyset \Leftrightarrow x + mu \in Fr(A)$
**that is: Fr (A) ∩ $L(x, u) \neq \emptyset$
d)* let ϵ>0 and x, y in A such that: ‖x−y‖≥diam (A) − ϵ.
*Suppose: x-y≠ 0
*let u=y − x, X= {t≥0|x+tu∈A} and t_0 = sup (X).
*We have: x+u=x+y-x=y∈ A ⇒ 1 ∈X⇒ sup (A) = t_0≥1 (because 1∈ X),
* By 3) c), we have: z=x+t_0u∈Fr (A).
* consider the half-line issued from x ∈A having for directory coefficient v=x-y: L(x, v) ={ x+tv, t≥ 0}
*By 3) c) Fr (A)∩L(x,v)≠ ∅ ⇒ $\exists t_1 \geq 0$ such that : z′=x+t_1v ∈Fr(A).
*we have: z′−z=(x+t_1v) − (x+t_0u) =x+t_1(x-y)-(x-t_0(x-y)) = $(t_1 + t_0)$(x−y) with : $t_1 + t_0 \geq 0 + 1 = 1$
*so: diam (A) ≥diam (Fr(A))≥‖z′−z‖=$(t_1 + t_0)$)‖x−y‖≥‖x−y‖≥diam(A)− ϵ.
*Finally, tending $\epsilon \to 0$, gives : diam(A) ≥diam (Fr(A))≥diam(A) i. e. : diam (A)=diam(Fr(A))

Definition 22: (of a connected topological space [72]) (X, τ) a topological space is called to be connected if:
$$\forall A \subset X: adh(A) = int(A) and\ A \neq \emptyset \Rightarrow A = X$$

Exemple: in the topological space $(\mathbb{R}, |.|)$ the intervals [a, b] are connected spaces for the induced topology.

Proposition 17: (minimal and maximal elements [61]) any non empty finite subset A of $\mathbb{N}$ has a smallest element noted min (A) and a greatest element noted max (A). We have: min $(A) \in A$ *and* $\max(A) \in A$.

Definition 23: (a directed set [78]) a **directed set** (or a **directed preorder** or a **filtered set**) $(E, \leq)$is a nonempty set E together with a reflexive (i.e.:$\forall x \in E\ x \leq x$) and transitive (i.e.$\forall x, y, z \in E: x \leq y\ and\ y \leq z \Rightarrow x \leq z$)binary relation $\leq$ (that is, a preorder), with the additional property that every pair of elements has an upper bound. That is, $\forall a, b \in E \exists c \in E$ such that: $a \leq c$ and $b \leq c$. A directed set's preorder is called a *direction*.

Definition 24: (a partially ordered set [79]) a **partially ordered set** (or a **poset**) formalizes and generalizes the intuitive concept of an ordering, sequencing, or arrangement of the elements of a set. A poset consists of a set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. The relation itself is called a "partial order."

Definition 25 :(the **inverse system** (or projective system) of groups and homomorphisms [80]). Let be a directed poset (not all authors require I to be directed). Let $(G_i)_{i\in I}$ be a family of groups and suppose we have a family of homomorphisms $f_{ij}: G_j \to G_i$for all (note the order) with the following properties:

$$\forall i \in I\ f_{ii} = \text{the identité application of the group } G_i\ ,$$

$$\forall i \leq j \leq k \in I\ f_{ik} = f_{ij} \circ f_{jk}$$

Then the pair : $((G_i)_{i\in I}, (f_{ij})_{i\leq j \in I})$ is called an inverse system of groups and morphisms over, and the morphisms are called the transition morphisms of the system.
We define the **inverse limit** of the inverse system as a particular subgroup of the direct product of the G_i's:

$$G=\varprojlim_{i\in I} G_i = \{\vec{g} = (g_i)_{i\in I} \in \prod_{i\in I} G_i\ , \forall i \leq j \in I\ \ a_i = f_{ij}(a_j)\}$$

Definition 26 : ($\mathbb{Q}_p$ the space of p −adic numbers [81]) for $p \in \mathbb{P}$ (i.e. is prime) we call :
(i)The p-adic valuation of $a \in \mathbb{Z}^*$, the number: $v_p(a)$ =the power of p in the decomposition of a as the finite product of powers of prime integers with $v_p(0) = +\infty$
(ii)The p-adic valuation of $\frac{a}{b} \in \mathbb{Q}$ (with: $a, b \in \mathbb{Z}^*$) is $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$
(iii) The p-adic absolute value is $|r|_p = p^{-v_p(r)}$ for $r \in \mathbb{Q}$ and $p \in \mathbb{P}$
(iv) The p-adic absolute value defines a topology such that:
(v)$U\ open \Leftrightarrow \forall x \in U \exists \epsilon(x) > 0$ Such that $B(x, \epsilon(x)) = \{y, |y - x|_p < \epsilon(x)\} \subset U$
(vi)$\mathbb{Z}_p$ The ring of p-adic integers is the completion of $(\mathbb{Z}, ||_p)$ i.e.: $\mathbb{Z} \subset \mathbb{Z}_p$ and adh $(\mathbb{Z}) = \mathbb{Z}_p$
(vii) $\mathbb{Q}_p$ The field of p-adic numbers is the completion of $(\mathbb{Q}, ||_p)$ i.e.: $\mathbb{Q} \subset \mathbb{Q}_p$ and adh $(\mathbb{Q}) = \mathbb{Q}_p$
(viii)$\mathbb{Z}_p = \{r \in \mathbb{Q}_p, v_p(r) \geq 0\}$

Proposition 18 : (1) We show that $\mathbb{Q}$ has only the two completion : $(\mathbb{R}, ||)$ and $(\mathbb{Q}_p, ||_p)$.
(2)$\mathbb{Q}_p$ is the arithmetic analog of $\mathbb{R}$.
(3) $|.|_p$ is an ultra-metric absolute value on $\mathbb{Q}_p$.
(4) In $\mathbb{Q}_p$ (i) $\lim_{n\to+\infty} p^n = 0$ (ii) $\lim_{n\to+\infty} \frac{1}{q^n}$ das not exist $\forall q \in \mathbb{P} - \{p\}$.
(5) In $\mathbb{Q}_p$: the series $\sum u_k$ converges$\Leftrightarrow$ $\lim_{k\to+\infty} u_k = 0$.
(6) the topological space $\mathbb{Q}_p$has a basis of open-closed subsets (we say that : it has dimension 0), so it is totally discret (i.e.$\forall x \in \mathbb{Q}_p$ $\{x\}$=its connected component).
(7) in $\mathbb{Q}_p$: $\lim_{k\to+\infty} u_k = u \neq 0 \Rightarrow \exists N \in \mathbb{N} \forall k \geq N\ |u_k|_p = |u_N|_p$ (i.e. it is constant since a certain order).

Definition 27 : (definition of the adelic ring [82]) (1) the profined completion $\hat{\mathbb{Z}}$ of integers is the projective limit (or the inverse limit) of the rings $\mathbf{Z}/n\mathbf{Z}$: $\hat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{Z}} \mathbf{Z}/n\mathbf{Z}$.

(2) By the Chinese rest theorem, it is isomorph to the product al the p-adic integers: $\hat{\mathbb{Z}} = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$. The right member being equipped with the product topology.

(3) the **adelic ring of integers is the** product :$A_{\mathbb{Z}} = \mathbb{R} \times \hat{\mathbb{Z}}$.
(4) the adelic ring of rationals is its extention of scalars to all the rationals, i.e. the tensor product :

$$A_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} A_{\mathbb{Z}}$$

Topologized in a way that $A_{\mathbb{Z}}$ is an open sub-ring of: $A_{\mathbb{Q}}$. Precisely a topological basis is given by the sets:

$$U \times \prod_{p \in S} v_p \times \prod_{p \notin S} \mathbb{Z}_p$$

Where U is an open subset of : $\mathbb{R}$, S is a finite set of prime integers and v_p is an open subset of : $\mathbb{Q}_p$.

(5) The adelic topology is more fine than that induced by the product topology of : $\mathbb{R} \times \prod_{p \in \mathbb{P}} \mathbb{Q}_p$. .
(6) More generally, the adelic ring of any field of algebraic numbers K is the tensoriel product :$A_K = K \otimes_{\mathbb{Z}} A_{\mathbb{Z}}$
(7)Topologized as the product of deg(K) copies of $A_{\mathbb{Q}}$, the adelic ring of the s rationals can also be defined as :

$$A_{\mathbb{Q}} = \mathbb{R} \times \{(a_p)_{p \in \mathbb{P}} \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p, \exists S \text{ finite} \subset \mathbb{P} \text{ such that: } \forall p \in \mathbb{P} \backslash S \ a_p \in \mathbb{Z}_p\}$$

PROOF OT THE RIEMANN HYPOTHESIS ACCORDING TO ROBIN CRITERION

Theorem: The Riemann Hypothesis is true.

Proof: (of the theorem)
*By the Robin Criterion it is sufficient to prove that: $\forall n \in \mathbb{N} \cap [5041, +\infty[$ we have: $\sum_{d|n} d \leq e^{\gamma} n \ln(\ln(n))$, where $\gamma = 0.5772$ denotes the Euler-Mascheroni constant.
*The Proof of the theorem will be deduced from the lemmas and claims below.
*Let:$n \in \mathbb{N} \cap [5041, +\infty[$.

First Step: if:$n \in \mathbb{N} \cap [5041, 5050]$.

Lemma1: We have: $\sum_{d|n} d \leq e^{\gamma} n \ln(\ln(n))$

Proof: (of lemma1)
*By proposition 4, and because $0.5772 < \gamma$, we have:
(1) $\sum_{d|5041} d = 5113 \leq e^{0.5772} \times 5041 \ln(\ln(5041)) = 19240.78 \ldots < 5041 e^{\gamma} \ln(\ln(5041))$
(2) $\sum_{d|5042} d = 7566 \leq e^{0.5772} \times 5042 \ln(\ln(5042)) = 19244.81 \ldots < 5042 e^{\gamma} \ln(\ln(5042))$
(3)$\sum_{d|5043} d = 6892 < e^{0.5772} \times 5043 \ln(\ln(5043)) = 19248.83 < 5043 e^{\gamma} \ln(\ln(5043))$
(4)$\sum_{d|5044} d = 8343 < e^{0.5772} \times 5044 \ln(\ln(5044)) = 19252.86 \ldots < 5044 e^{\gamma} \ln(\ln(5044))$
(5)$\sum_{d|5045} d = 6060 < e^{0.5772} \times 5045 \ln(\ln(5045)) = 19256.88 \ldots < 5045 e^{\gamma} \ln(\ln(5045))$
(6)$\sum_{d|5046} d = 9611 < e^{0.5772} \times 5046 \ln(\ln(5046)) = 19260.91 \ldots < 5046 e^{\gamma} \ln(\ln(5046))$
(7)$\sum_{d|5047} d = 5928 < e^{0.5772} \times 5047 \ln(\ln(5047)) = 19264.94 \ldots < 5047 e^{\gamma} \ln(\ln(5047))$
(8)$\sum_{d|5048} d = 7178 < e^{0.5772} \times 5048 \ln(\ln(5048)) = 19268.96 \ldots < 5048 e^{\gamma} \ln(\ln(5048))$
(9)$\sum_{d|5049} d = 8640 < e^{0.5772} \times 5049 \ln(\ln(5049)) = 19272.99 \ldots < 5049 e^{\gamma} \ln(\ln(5049))$
(10)$\sum_{d|5050} d = 6111 < e^{0.5772} \times 5050 \ln(\ln(5050)) = 19277.01 \ldots < 5050 e^{\gamma} \ln(\ln(5050))$

Second step: $n \in \mathbb{N} \cap [5051, +\infty[$.

Lemma2: We have: $\sum_{d|n} d \leq e^{\gamma} n \ln(\ln(n))$.

Proof: (of lemma2)
The proof of lemma2 will be deduced from the claims below.

Claim1: $\exists m \geq 676$ such that $n \in [p_m, p_{m+1}[$.

Proof: (of claim1)
*By proposition 5, we have $5051 = p_{676} \Rightarrow \mathbb{N} \cap [5051, +\infty[= \mathbb{N} \cap \bigcup_{m=676}^{+\infty} [p_m, p_{m+1}[= \bigcup_{m=676}^{+\infty} (\mathbb{N} \cap [p_m, p_{m+1}[)$.
*So:$\exists m \geq 676$ such that: $n \in \mathbb{N} \cap [p_m, p_{m+1}[$

Definition20: Consider for $m \geq 676$, the sets:

*$A_m = \{t \in [p_m, p_{m+1}],\ \sum_{E\left(\frac{-E(-t)}{d}+\frac{1}{2}\right)=\frac{-E(-t)}{d}} d \le -E(-t)e^{\gamma}\ln(\ln(-E(-t)))\}$

*And $B_m = A_m^c$

Remark: The function $f(t) = t\ln(\ln(t))$ is a strictly increasing continuous function because $f'(t) = \ln(\ln(t)) + \frac{1}{\ln(t)} > 0$ for $t \ge 5051$

Claim2: We have:
$\forall t \in \mathbb{R}_+^*\ d| -E(-t) \Leftrightarrow \frac{-E(-t)}{d} \in \mathbb{N}^* \Leftrightarrow E\left(\frac{-E(-t)}{d} + \frac{1}{2}\right) = \frac{-E(-t)}{d}$, where $E(x)$ denotes the integer part of the real x.

Proof: (of claim2)
The result follows by definition 5 and proposition 6.

Claim3: We have:
$B_m = \{t \in [p_m, p_{m+1}]$ such that: $\sum_{E\left(\frac{-E(-t)}{d}+\frac{1}{2}\right)=\frac{-E(-t)}{d}} d > -E(-t)e^{\gamma}\ln(\ln(-E(-t)))\}$

Proof: (of claim3)
The result is evident by taking the negation of the relation defining the set A_m.

Claim4: if $m \ge 676$, we have:
(1) $p_m \in A_m$ with $p_m = \min(A_m)$
(2) $p_{m+1} \in A_m$ with $p_{m+1} = \max(A_m)$
(3) So: $A_m \ne \emptyset$

Proof: (of claim4)
(1)*p_m being prime, we have: $D(p_m) = \{1, p_m\}$
*So, we have: $\sum_{d|p_m} d = 1 + p_m$
*We have: $0.5772 < \gamma \Rightarrow 1 < 14199{,}7859\ldots = 5041(e^{0.5772}\ln(\ln(5041) - 1) < p_m(e^{\gamma}\ln(\ln(p_m)) - 1)$
*So:$p_m e^{\gamma}\ln(\ln(p_m)) > p_m + 1 = \sigma(p_m)$
*That is: $p_m \in A_m$ because: $-E(-p_m) = p_m$
(2) The result follows as in (1) of claim4.
(3) The result follows by the assertions (1) and (2) of claim 4.

Claim5: We have: $p_{m+1} \in \operatorname{int}(A_m)$

Proof: (of claim5)
*Suppose contrarily that: $p_{m+1} \notin \operatorname{int}(A_m)$ i.e $p_{m+1} \in (\operatorname{int}(A_m))^c = adh(A_m^c)$
*So: $\exists (t_k)_k \subset A_m^c = B_m \subset [p_m, p_{m+1}]$ such that: $\lim_{k\to+\infty} t_k = p_{m+1}$ with: $t_k \le p_{m+1}$ for any k
*We have: $t_k = p_{m+1} - h_k \in B_m$ with $\lim_{k\to+\infty} h_k = 0$ and$1 > h_k > 0$
*We have: $0 \le E(h_k) \le h_k \Rightarrow \lim_{k\to+\infty} E(h_k) = 0$
*We have: $c_k, d_k \le -E(-t_k) = c_k d_k \le p_{m+1}$ with $c_k, d_k \in \mathbb{N}^*$
*So: $\lim_{k\to+\infty} d_k = d$ and $\lim_{k\to+\infty} c_k = c$ with $\lim_{k\to+\infty} -E(-t_k) = cd$
*We have: $\frac{-(E(-p_{m+1}))-E(h_k)}{d_k} \to \frac{-(E(-p_{m+1})}{d} \in \mathbb{N} \Rightarrow$

$$\lim_{k\to+\infty} E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k} + \frac{1}{2}\right) = E\left(\frac{-E(-p_{m+1}) - \lim_{k\to+\infty} E(h_k)}{\lim_{k\to+\infty} d_k} + \frac{1}{2}\right) = E\left(\frac{-E(-p_{m+1})}{d} + \frac{1}{2}\right)$$

*So:
$\forall k \in \mathbb{N}\ \sum_{E\left(\frac{-E(-t_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-t_k)}{d_k}} d_k = \sum_{E\left(\frac{-E(-(p_{m+1}-h_k))}{d_k}+\frac{1}{2}\right)=\frac{-E(-(p_{m+1}-h_k)}{d_k}} d_k = \sum_{E\left(\frac{-E(-p_{m+1}+h_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1}+h_k)}{d_k}} d_k$
$=\sum_{E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})-E(h_k)}{d_k}} d_k > -E(-t_k)e^{\gamma}\ln(\ln(E(-t_k)))$
$\ge E(-E(-t_k)e^{\gamma}\ln(\ln(-E(-t_k)))) = E((-E(-p_{m+1}) - E(h_k))e^{\gamma}\ln(\ln(-E(-p_{m+1}) - E(h_k))$
$=E(-E(-p_{m+1})e^{\gamma}\ln(\ln(-E(-p_{m+1}))$

*So: $\sum_{E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})-E(h_k)}{d_k}} d_k$ and $E(-E(-t_k)e^{\gamma}\ln(\ln -E(-t_k))))$ being integers, we have:

$\sum_{E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})-E(h_k)}{d_k}} d_k \geq E(-E(-t_k)e^{\gamma}\ln(\ln(-E(-t_k))))$ +1

$=E(-E(-p_{m+1})e^{\gamma}\ln(-E(-p_{m+1})) + 1$

*So:

$\lim_{k\to+\infty}\sum_{E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})-E(h_k)}{d_k}} d_k = \sum_{\lim_{k\to+\infty} E\left(\frac{-(E(-p_{m+1})-E(h_k)}{d_k}+\frac{1}{2}\right)=\lim_{k\to+\infty}\frac{-E(-p_{m+1})-E(h_k)}{d_k}} \lim_{k\to+\infty} d_k$

$=\sum_{E\left(\frac{-E(-p_{m+1})-\lim_{k\to+\infty}E(h_k)}{\lim_{k\to+\infty}d_k}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})-\lim_{k\to+\infty}E(h_k)}{\lim_{k\to+\infty}d_k}} d = \sum_{E\left(\frac{-E(-p_{m+1})}{d}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})}{d}} d$

$\geq E(-E(-p_{m+1})e^{\gamma}\ln(-E(-p_{m+1})) + 1 > -E(-p_{m+1})e^{\gamma}\ln(-E(-p_{m+1}))$

*But: $\sum_{E\left(\frac{-E(-p_{m+1})}{d}+\frac{1}{2}\right)=\frac{-E(-p_{m+1})}{d}} d > -E(-p_{m+1})e^{\gamma}\ln(-E(-p_{m+1}))$ means that: $p_{m+1} \notin A_m$

*This contradicting the assertion (2) of claim 4, we have well: $p_{m+1} \in \text{int}\,(A_m)$

Claim6: Fr (A_m) =adh (A_m) $-$int (A_m) is empty or is a finite set containing only integer numbers.

Proof: (of claim 6)

*Let t ∈Fr (A_m) =adh $(A_m) \cap$ adh (A_m^c)

*We have: $t = \lim_{k\to+\infty} t_k = \lim_{k\to+\infty} s_k$ with: $t_k \in A_m$ and $s_k \in A_m^c$

*We have:

**$\sum_{E\left(\frac{-E(-t_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-t_k)}{d_k}} d_k \leq -E(-t_k)e^{\gamma}\ln(\ln(-E(-t_k)))$

**$\sum_{E\left(\frac{-E(-s_k)}{d_k}+\frac{1}{2}\right)=\frac{-E(-s_k)}{d_k}} d_k > -E(-s_k)e^{\gamma}\ln(\ln(-E(-s_k)))$

*If $t \notin \mathbb{N}$, we have: $\lim_{k\to+\infty} E(-t_k) = \lim_{k\to+\infty} E(-s_k) = E(-t)$

* Working with integers, we have: $\exists N$ such that: $\forall k \geq N\ E(-t_k) = E(-s_k) = E(-t)$

*So: tending $k \to +\infty$ in the precedent relations, we have:

** $\sum_{E\left(\frac{-E(-t)}{d}+\frac{1}{2}\right)=\frac{-E(-t)}{d}} d \leq -E(-t)e^{\gamma}\ln(\ln(-E(-t)))$

**$\sum_{E\left(\frac{-E(-t)}{d}+\frac{1}{2}\right)=\frac{-E(-t)}{d}} d > -E(-t)e^{\gamma}\ln(\ln(-E(-t)))$

*This meaning that: $t \in A_m \cap A_m^c = \emptyset$, which is impossible, we have well:

$$t \in Fr(A_m) \Rightarrow t \in \mathbb{N}$$

*Finally: Fr $(A_m) \subset [p_m, p_{m+1}] \cap \mathbb{N}$ finite ⇒ Fr (A_m) is finite and contains only integers.

Claim7: If A is a non empty bounded subset of $\mathbb{R}$, having min $(A) \in A$ and max $(A) \in A$, then:

$$diam(A) = \max(A) - \min(A)$$

Proof: (of claim7)

*By definition of min (A) and max (A), we have:

$$\forall x, y \in A \quad \min(A) - \max(A) \leq x - y \leq \max(A) - \min(A)$$

*That is, by definition 1, $\forall x, y \in A$:$|x - y| \leq \max(A) - \min(A)$

*So: $diam(A) = sup_{x,y\in A}|x - y| \leq \max(A) - \min(A)$

*But: $\max(A) \in A$ and $\min(A) \in A \Rightarrow \max(A) - \min(A) = |\max(A) - \min(A)| \leq diam(A) = sup_{x,y\in A}|x - y| \leq \max(A) - \min(A)$

*The result follows.

Claim8: We have: $diam(A_m) = p_{m+1} - p_m$

Proof: (of claim8)

*By claim 4: $p_m = \inf(A_m) \in A_m$ and $p_{m+1} = \sup(A_m) \in A_m$

*So, by definition of $diam(A_m)$ and claim 7, we have: $diam(A_m) = p_{m+1} - p_m$

Claim 9: if $Fr(A_m) \neq \emptyset$, we have: $diam(Fr(A_m)) = p_{m+1} - p_m$

Proof: (of claim 9)

The result follows by combination of proposition 15 and claim 8.

Claim10: we have: $Fr(A_m) = \emptyset$

Proof: (of claim10)
*If $Fr(A_m) \neq \emptyset$, by claim 9, we have: $diam(Fr(A_m)) = p_{m+1} - p_m$
*But, by claim 6, $Fr(A_m)$ is a finite set containing only integers
*So, we have necessarily: p_m and $p_{m+1} \in Fr(A_m) = adh(A_m) - int(A_m)$
*Indeed: $Fr(A_m)$ being a non empty (by the absurd hypothesis) finite subset of $\mathbb{N} \cap [p_m, p_{m+1}]$, it has, by proposition 19, a min $(Fr(A_m)) = p_m + h \in Fr(A_m)$ and a $\max\big(Fr(A_m)\big) = p_{m+1} - k \in Fr(A_m)$ with $h, k \geq 0$
*So, by claim 7 and claim 9: diam $(Fr(A_m)) = \max(Fr(A_m)) - \min(Fr(A_m)) = p_{m+1} - k - (p_m + h) = p_{m+1} - p_m - k - h = p_{m+1} - p_m \Rightarrow k + h = 0 \Rightarrow k = h = 0$
*But, by claim5, we have:$p_{m+1} \in int(A_m)$
*So, having obtained the impossible relation: $p_{m+1} \in Fr(A_m) \cap int(A_m) = \emptyset$, we have well $Fr(A_m) = \emptyset$

Claim 11: we have: $A_m = [p_m, p_{m+1}]$
Proof :(of claim11)
*By proposition 14 and claim 10, we have:

$$\begin{cases} adh(A_m) = int(A_m) \cup Fr(A_m) \\ Fr(A_m) = \emptyset \end{cases} \Rightarrow adh(A_m) = int(A_m)$$

*So, by claim 4 and definition 18, we have:

$$\begin{cases} A_m \subset [p_m, p_{m+1}] \\ [p_m, p_{m+1}] connected \\ adh(A_m) = int(A_m) \\ A_m \neq \emptyset \end{cases} \Rightarrow A_m = [p_m, p_{m+1}]$$

RETURN TO THE PROOF OF THE THEOREM
*But combination of claim 1 and claim 11, we have: $n \in [p_m, p_{m+1}] = A_m$.
*So:$\forall n$ integer ≥ 5051 $\exists m$ integer ≥ 676 such that: $n \in A_m$.
*So by combination of lemmas 1 and 2, we have:$\forall n$ integer ≥ 5041: $\sum_{d|n} d \leq ne^{\gamma} \ln(\ln(n))$.
*This is the Robin inequality equivalent to the Riemann Hypothesis.
*So, the Riemann Hypothesis is showed.

SOME CONSEQUENCES
Solving RH yields even the solutions up to 500 other unsolved problems.
In 1927, Edmund Landau (1877-1938) showed in [38] that when the Riemann Hypothesis is admitted a lot of consequences will follow.

1) The RH is applicable not only in pure mathematics, but also in many other scientific fields such as physics and engineering (see G. Lachaud [34]).
*Youri Manine said in his book "mathematics and physics": "the most profound ideas of the number theory present a considerable analogy with that of the modern theoretical physics. Such as the quantum mechanics, the number theory gives relation models between the discrete and the continuous, and valorizes the hidden symmetries ".

*We can summaries [34], about: "RH and physics", as follows:
(i) By the arithmetical fundamental theorem: any natural integer is composed by prime integers. So the prime integers are for mathematicians as are the elements of the Mendeleyev table for chemists or the elementary particles for physicists. Euclid, in his "elements", calls them "protons numbers ".
(ii) The prime integers have a stochastic distribution.
(iii)The distribution of prime integers is as the distribution of molecules in a perfect gas.
(iv)If we write the zeros of the zeta function: $\rho_n = \frac{1}{2} + i\gamma_n$, we have: RH true$\Leftrightarrow$ $\forall n \in \mathbb{N}$ $\gamma_n \in \mathbb{R}$. So, having showed in the present paper that RH is true, we have well: $\forall n \in \mathbb{N}$ γ_n
(v) the most vibratory systems, as the light, the sound and waves, can be expressed as a superposition of basic signals: $f(u) = \sum_{k=1}^{+\infty} a_k \cos(\omega_k u)$: in acoustic the proper states are the pure sounds, in optic they are the

monochromatic waves.
(vi) Riemann, himself, has observed that the deviations, relatively to the low in $\frac{1}{\ln(x)}$ of the prime numbers density, are governed by a function having the wave form: $f(u) = \sum_{n=1}^{+\infty} \cos(\gamma_n u)$
(vii) So the γ_n are, as has noted M. Berry and J.P Keating, the harmonics of the prime integers music.
(viii) In such a system only the amplitudes exchange.
(ix) The exchange between two states is realized by a linear operator.
(x) The operators of concrete physical phenomena have real spectrum frequencies.
(xi) If we can find an operator A having the spectrum:$\sigma(A) = \{\gamma_n, n \in \mathbb{N}^*\} \subset \mathbb{R}$, the RH will be proved.
(xii) Such operator exists it is the Polya-Hilbert operator
(xiii) Berry and Keating supposed that the zeros of the zeta function are exactly the proper values (energy levels) of a hypothetic mechanical quantum system having chaotic trajectories.
(xiv) Bernard Julia expressed the zeta function as the function of the thermodynamically partition of a certain "perfect abstract gas" from prime numbers
(xv) Alain Connes and Jean Binoit Bost constructed a quantum dynamical system having, for which the partition function is the zeta function. Their construction uses the Adelic rings (See definition 27).
(xvi) Alain Connes constructed for any function L, an operator A, on a space of functions having adelic variables such that the spectrum of A is: $\sigma(A) = \{\gamma_n \in \mathbb{R}, L\left(\frac{1}{2} + \gamma_n i\right) = 0\}$. Connes said "The RH will be proved if this spectrum contains all the zeros". So, the RH being proved in the present paper: $\sigma(A)$ contains all the zeros.

2) Because the assertions cited in the below corollaries are equivalent to the RH, our main theorem gives the following results:

Corollary 1: (Schoenfeld inequality)$\forall s \geq 2657 \quad |\pi(s) - \int_0^s \frac{dt}{\ln(t)}| \leq \frac{\sqrt{s}\ln(s)}{8\pi}$

That is: $\pi(s) = \int_0^s \frac{dt}{\ln(t)} + O(\sqrt{s}\ln(s))$

Proof: See Bombieri [8], Borwein [10] and Ghanim [22]

Corollary 2: if $\mu(n) = \begin{cases} 0 \; if \; n \text{ has a square factor} \\ 1 \; if \; n = 1 \\ (-1)^k \text{ when n includs a product of k distict primes} \end{cases}$ is the Mobius function and $M(x) = \sum_{n\leq x} \mu(n)$ is the Mertens function, we have:$\forall \epsilon > 0 \;\; M(x) = O(x^{\frac{1}{2}+\epsilon})$

Proof: See Titchmarsh [52]

Corollary 3: (Lagarias inequality) we have: $\sum_{d|n} d \leq \sum_{k=1}^{n} \frac{1}{k} + e^{\sum_{k=1}^{n}\frac{1}{k}} \sum_{k=1}^{n} \frac{1}{k}$

Proof: See Lagarias [36]

Corollary 4: If the Dirichlet's eta function is:$\eta(s) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^s} = (1 - 2^{1-s}) \sum_{k=1}^{+\infty} \frac{1}{k^s}$, we have:

$$\eta(s) = 0, Re(s) \in]0,1[\Rightarrow Re(s) = \frac{1}{2}$$

Corollary 5: if $\omega(k)$ denotes the number of prime factors of the integer k and $\lambda(k) = (-1)^{\omega(k)}$ is the Liouville function we have: $\forall \epsilon > 0 \quad \lim_{n\to+\infty} \frac{\sum_{k=1}^{n} \lambda(k)}{n^{\epsilon+\frac{1}{2}}} = 0$.

Corollary 6: if $F_n = \left\{\frac{a}{b} \in \mathbb{Q} \cap [0,1], 0 \leq a \leq b \leq n, \gcd(a,b) = 1\right\} = \{F_n(k), k = 1, 2, \dots, m(n)\}$ denotes the Farey's series and $m(n) = card\, F_n$, we have: $\forall \epsilon > 0 \quad \sum_{k=1}^{m(n)} \left|F_n(k) - \frac{k}{m(n)}\right| = O(n^{\epsilon+\frac{1}{2}})$

Proof: See Edwards [20]

Remark: A result of Mertens asserts that: $m(n) = \sum_{k=1}^{n} \varphi(k) = \frac{n^2}{\pi^2} + O(n\ln(n))$ with $\varphi(n) = \sum_{m\leq n, \gcd(m,n)=1} 1$ is the Euler function.

Corollary7: (1976 Shapiro inequality) if $\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ is the Jacobi function, $\psi(x) = \frac{\theta(x)-1}{2} = \sum_{n\geq 1} e^{-\pi n^2 x}$ and $\delta(n) = e^{\int_0^n \psi(t)dt}$, we have: $\forall n \in \mathbb{N} \quad \left|\sum_{k=1}^{\delta(n)} \frac{1}{k} - \frac{n^2}{2}\right| < 36n^3$

Corollary 8: $\forall s \in \{s \in \mathbb{C}, Res(s) \in]0, \frac{1}{2}[\}$ we have: $\zeta'(s) \neq 0$

Proof: See Speiser [51].

Corollary 9: (Lagarias inequality) if: $\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, we have:

$$\forall s \in \{s \in \mathbb{C}, Res(s) \in]0,\frac{1}{2}[\} \quad Re\left(\frac{\xi'(s)}{\xi(s)}\right) > 0$$

Proof: See Lagarias [35].

Corollary 10: if:$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}(s^n \ln(\xi(s)))|s = 1$, we have: $\forall n \geq 1\ \lambda_n > 0$

Proof: See Li [39].

Corollary11: We have: $\forall n \geq 1\ \ \lambda_n|_{s=1} = \sum_{\rho \text{ the complex zeros of } \zeta}(1-\left(1-\frac{1}{\rho}\right)^n)$.

Proof: See Bombieri -Lagarias [9].

Corollary12: we have: $\sum_{k=1}^{+\infty}\frac{(-x)^k}{k!\zeta(2k+1)} = O(x^{-\frac{1}{4}})$ as x tends to infinity

Proof: See Hardy-Littlewood [33].

Corollary 13: for $\sigma \in]\frac{1}{2},1[$, the integral equation: $\int_{-\infty}^{+\infty}\frac{e^{-\sigma y}\varphi(y)}{e^{e^{x-y}}+1}dy = 0$ has no bounded solution other than the trivial one $\varphi(y) = 0$

Proof: See Salem [50].

Corollary 14: (Volchkov equality) we have:$\int_0^{=\infty}\int_{\frac{1}{2}}^{+\infty}\frac{(1-12t^2)\ln(\zeta(\sigma+it))dtd\sigma}{(1+4t^2)^3} = \frac{\pi(3-\gamma)}{32}$, where $\gamma$is the Euler-Mascheroni constant

Proof: See Volchkov [56].

Corollary 15: if Li^{-1} denotes the inversion of the function $Li(n) = \int_2^n\frac{dt}{\ln(t)}$ and $g(n)$ denotes the maximal order of elements of the symmetric group S_n of order n, we have: $\ln(g(n)) < \sqrt{Li^{-1}(n)}$.

Proof: See Massias- Nicolas-Robin [41].

Corollary 16: We have: $I = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\ln(|\zeta(s)|)dt}{|s|^2} = \sum_{Re(\rho)>\frac{1}{2},\ \rho \text{ being a zero of } \zeta}\ln\left(\frac{\rho}{1-\rho}\right) = 0$ where $s = \sigma + it$

Proof: See Balazard-Saias-Yor [3].

Corollary 17: (Nonpositivity of the Bruijn-Newman constant) if $\Xi(iz) = \frac{1}{2}\left(z^2-\frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta(z+\frac{1}{2})$ is the Xi Bruijin-Newman function, the signal $\Phi(t) = \sum_{n=1}^{+\infty}(2\pi^2n^2e^{9t}-3\pi n^2e^{5t})e^{-\pi n^2e^{4t}} = \frac{\Xi(\frac{z}{2})}{8}$ (with: $t \in [0,+\infty[$) and: $H(\lambda,z) = \mathcal{F}_t[\Phi(t)e^{\lambda t^2}](z)$, (wit:: $\lambda \in \mathbb{R}, z \in \mathbb{C}$), is the Fourier transform of $\Phi(t)e^{\lambda t^2}$, then:

*$\exists\Lambda$ such that: $H(\lambda,z) = 0 \Rightarrow (z \in \mathbb{R} \Leftrightarrow \lambda \geq \Lambda)$

*RH $\Leftrightarrow \Lambda \leq 0$

Remark: The best bound known for Λ is $\Lambda > -2.7.10^{-9}$ obtained by Odlyzko [43].

Proof: See Borwein-Choi-Rooney-Weirathmueller [10], Csordas-Odlyzko-Smith-Varga [17], Csordas-Smith-Varga [18] and Newman [42].

Corollary 18: (Redheffer Criterion) if $R_n = (R_n(i,j))$ is the $n \times n$ Redheffer matrix defined by:

$$R_n(i,j) = \begin{cases}1 & if\ j = 1\ or\ if\ i\ divides\ j\\ 0 & otherwise\end{cases}$$

We have: $\forall\epsilon > 0\ \ \det(R_n) = \sum_{k=1}^{n}\mu(k) = O(n^{\frac{1}{2}+\epsilon})$, where

$$\mu(n) = \begin{cases}0 \text{ when n has a square factor}\\ 1 \text{ when n} = 1\\ (-1)^k \text{ if when n including a product of k distinct primes}\end{cases}$$ is the Mobius function.

Proof: See Redheffer [43].

Corollary 19: if R_n is the Redheffer matrix of order$n$, I_n is the identity matrix of order n, G_n is a graph having the adjacency matrix $B_n = R_n - I_n$ and $\overline{G_n}$ is the graph extending the graph G_n adding a loop at node 1 of G_n. Then, we have:$\forall\epsilon > 0$ |card {even cycles inside $\overline{G_n}$}-card{odd cycles inside $\overline{G_n}$}|=$O(n^{\frac{1}{2}+\epsilon})$

Card E denoting the number of the finite set E.

Proof: See Redheffer [46].

Corollary 20: $\forall y > 0\exists m(y)$ ergodic measures supported on closed orbits of period $\frac{1}{y}$ of the horocycle flow on the space $M = PSL_2(\mathbb{Z})/PSl_2(\mathbb{R})$ such that: $\forall\epsilon > 0\ \forall f\ smooth\ function\ on\ M,$ we have:

$$\int_M f dm(y) = O(y^{\frac{3}{4}-\epsilon}) \text{ when } y \to 0$$

Proof: See Verjovsky [54].

Corollary21: if

$*\ C_c^r(\mathbb{R}^*) = \{f: \mathbb{R}^* \to \mathbb{C},$ r twice differentiable with compact support$\}$,

*$\varphi(n) = \sum_{m\leq n,\gcd(m,n)=1} 1$, is the Euler function,
*$m_y(f) = \sum_{n\in\mathbb{N}} y\varphi(n)f(y^{\frac{1}{2}}n)$,
*and $m_0(f) = \int_0^{+\infty} uf(u)du$
Then:$\forall\epsilon > 0\forall f \in C_c^r(\mathbb{R}^*)|_{r=2}$ we have: $m_y(f) = m_0(f) + O(y^{\frac{3}{4}-\epsilon})$ when $y \to 0$
Proof: See Verjovsky [53].
Corollary 22: if$L^2(]0,1[) = \{f:]0,1[\to \mathbb{R}, \int_{-\infty}^{+\infty}(|f(t)|)^2\, dt < +\infty\}$, {x} denotes the fractional part of the real x and: $\rho_\alpha(t) = \left\{\frac{\alpha}{t}\right\} - \alpha\{\frac{1}{t}\}$, then: the closed linear span of $\{\rho_\alpha(t), 0 < \alpha < 1\}$ =$L^2(]0,1[)$
Proof: See Balazard-Saias [4].
Corollary 23: if A is the Hilbert-Schmidt operator defined on $L^2(]0,1[)$ by $[A(f)]\,(\theta) = \int_0^1 f(x)\{\frac{\theta}{x}\}\,dx$, then A is injective.
Proof: See Alkantara, J.-Bode [1].
Corollary 24: $\Xi(t) = \xi\left(\frac{1}{2} + it\right) = \sum_{n=1}^{+\infty} c_n t^n = 0 \Rightarrow t \in \mathbb{R}$
Proof: See Alter [2].
Corollary 25: (Alter property) consider the entire function: $f(z) = \sum_{n=1}^{+\infty} a_n z^n$ then $f(-t^2) = \sum_{n=1}^{+\infty}(-1)^n a_n t^{2n}$ is an entire function of order$\frac{1}{2}$. If t_0 is a real zero of the function $\Xi(t) = \xi\left(\frac{1}{2} + it\right)$, then $z_0 = -t_{à}{}^2$ is a negative zero of f and all the zeros of f are negative.
Proof: See Alter [2].
Corollary 26: (Alter and Ramanujan property) if $\tau(n)$ is the Ramanujan arithmetical function, then the associated Dirichlet series: $z(s,\tau) = \sum_{n=1}^{+\infty} \frac{\tau(n)}{n^s}$ is absolutely convergent and is regular for $\sigma > \frac{13}{2}$.
Remarks: (1) Wilton asserts, in [83], that: $z(s,\tau)$ is an entire function holding:

$$(2\pi)^{-s}\Gamma(s)z(s,\tau) = (2\pi)^{s-12}\Gamma(12-s)z(12-s,\tau)$$

(2) On the other hand, its simple zeros are at: $s = 0, -1, -2, \ldots$
(3)Alter (see [2]) defines: $\xi(s,\tau) = (2\pi)^{-s}\Gamma(s)z(s,\tau)$ and states that the poles of $\Gamma(s)$ are canceled by the zeros of $z(s,\tau)$ so that $\xi(s,\tau)$ should be an entire function of s such that $\xi(12-s,\tau) = \xi(s,\tau)$ meaning that $\xi(6-it,\tau) = \xi(6+it,\tau)$ and we have symmetry about the line: $\sigma = Re(s) = 6$.
(4) If we put $s = 6 + iz \; or \; s = 6 + it$ in the relation given in (3) above, we have:

$$\Xi(z,t) = \xi(6+it,\tau) = (2\pi)^{-6-it}\Gamma(6+it)z(6+it,\tau)$$

So: all the zeros of $\Xi(z,t)$ are real ⇔all the zeros of $z(s,\tau)$ lie on the line $s = 6 + iz$
(5) Wilton (see [83]) showed that $z(s,\tau)$ has an infinite number of zeros on the line $\sigma = Re(s) = 6$
(6) We have: all the zeros of $z(s,\tau)$ in the critical strip $\frac{11}{2} < \sigma = Re(s) < \frac{13}{2}$ lie on the line $\sigma = Re(s) = 6$
Proof: See Alter [2].
Corollary 27: if Laguerre-Polya class is defined as the collection of all real functions:

$$f(x) = Ce^{-\alpha x^2+\beta x}x^n\prod_{j=1}^{\omega}(1-\frac{x}{x_j})e^{\frac{x}{x_j}}$$

With: (i) $\omega \leq +\infty$ (ii)$\alpha \geq 0, \beta, C$ denote real numbers (iii) $n \in \mathbb{N}.^*$
(iv) x_j are nonzero reals such that: $\sum_{j=1}^{\omega}\frac{1}{x_j^2} < +\infty$.
Then: the function $\xi(x)$ belongs to the Laguerre-Polya class.
Proof: See Polya [44], Salem [50] and Csordas-Varga [16].
Corollary 28: if $\gamma_k \in \mathbb{R}^+$ satisfy the Turan inequality: "$\forall k \in \mathbb{N}^*\ \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$", and if the n^{th} Jensen polynomial corresponding to the function $f(x) = \sum_{k=0}^{+\infty}\gamma_k\frac{x^k}{k!}$ is defined as: $g_n(t) = \sum_{k=0}^{n}\gamma_k t^k C_n^k$ for:$n \in \mathbb{N}$, then the Jensen polynomials corresponding to the Riemann zeta function ζ are hyperbolic at their points of symmetry. Recall that a polynomial of real coefficients is hyperbolic if all its zeros are real.
Remark: (1) After Polya [44], we can say that: all Jensen polynomials associated with the sequence of Taylor coefficients $\gamma(n)$ are hyperbolic. Recall that: $\gamma(n)$ are defined by: $(4z^2-1)\,\Lambda\left(z+\frac{1}{2}\right) = \sum_{n=0}^{\infty}\gamma(n)\frac{z^{2n}}{n!}$.
(2)If the Jensen polynomial of degree d and shift n is:$J_\alpha^{d,n}(X) = \sum_{j=0}^{d} c_d^j\,\alpha(n+j)X^j$, with: $\alpha = (\alpha(0), \alpha(1), \ldots)$ is an arbitrary sequence of real numbers, then: $\forall n, d \in \mathbb{N}$ the polynomials $J_\alpha^{d,n}(X)$ are hyperbolic.
Proof: See Csordas-Craven 1989[15].
Corollary29: the necessary and sufficient conditions for holding real zeros for the function:$F(z) = \int_0^{+\infty}\Psi(t)\cos(zt)\,dt$ are:

(i)$\forall x, y \in \mathbb{R}$: $\int_0^\infty \int_0^\infty \Psi(\alpha)\Psi(\beta)e^{i(\alpha+\beta)x}e^{(\alpha-\beta)y}(\alpha-\beta)^2 d\alpha d\beta \geq 0$
(ii) $\forall x \in \mathbb{R} \forall n \in \mathbb{N}$: $\int_0^\infty \int_0^\infty \Psi(\alpha)\Psi(\beta)e^{i(\alpha+\beta)x}(\alpha-\beta)^{2n} d\alpha d\beta \geq 0$
(iii) $\forall x \in \mathbb{R} \forall n \in \mathbb{N}$: $\int_0^\infty \int_0^\infty \Psi(\alpha)\Psi(\beta)(x+i\alpha)^n(x+\beta i)^n(\alpha-\beta)^2 d\alpha d\beta \geq 0$
Remark: Since F belong to Laguerre-Polya class functions, F has real zeros.
Proof: See Csordas-Craven 1989[15].

Corollary 30: (Lindelof hypothesis) we have: $\forall \epsilon > 0\ \zeta\left(\frac{1}{2}+it\right) = O(t^\epsilon)$

Proof: See Titchmarsh [52].

Corollary 31: (the Weil positivity statement) for all nice test functions f, we have: $W^{(1)}(f * \hat{\bar{f}}) \geq 0$
Recall that:
(1)Nice test functions are functions $f:]0, +\infty[\to \mathbb{C}$ piecewise on C^2 (C denoting a small curve drawn around a zero of the function f), having compact support and the averaging property:

$$f(x) = \frac{1}{2}(\lim_{t\to x^-} f(t) + \lim_{t\to x^+} f(t))$$

(2) The Mellin transform of $f(s)$ is defined as: $\mathcal{M}[f](s) = \int_0^{=\infty} f(x)x^{s-1}dx$
(3) The convolution operation associated with Mellin transform is defined as:

$$(f * g)(x) = \int_0^{+\infty} f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}$$

(4) Note that we have: $\mathcal{M}[f * g](s) = \mathcal{M}[f](s)\mathcal{M}[g](s)$
(5) If $f(x) = u(x) + iv(x)$, we have: $\bar{f}(x) = u(x) - iv(x)$
(6) We have: $\hat{f}(x) = \frac{1}{x} f(\frac{1}{x})$
(7) We have: $\mathcal{M}[\hat{f}](s) = \mathcal{M}[f](1-s)$
(8) For all nice test functions, define $W^{(1)}(f) = \sum_{\rho \text{ zero of } \xi} \mathcal{M}[f](\rho)$ with:

$$\xi(s) = \frac{S}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{S}{2}\right)\zeta(s)$$

(9) For two "nice" functions f and g, we can define the following intersection product:

$$< f, g >= W^{(1)}(f * g)$$

Proof: See Lagarias 2006 [37].

Corollary 32: We have: $\forall \epsilon > 0 \forall\ f \in C_{00}^\infty(S\mathbb{H}/G)\ \frac{1}{t}\int_{\gamma_t} f(z)d\nu_t(z) = \frac{\int_{\mathbb{H}/\Gamma} f(z)d\mu(z)}{Vol(\mathbb{H}/G)} + O(t^{\frac{3}{4}+\epsilon})$ when $t \to 0$

Recall that:
(1)$\mathbb{H}$ Is the hyperbolic plane.
(2) $G = PSL(2, \mathbb{Z})$ A group acting on $\mathbb{H}$.
(3) We can generate the group G by isometrics: $z \mapsto z+1$ and: $z \mapsto -\frac{1}{z}$ of the half-plane model of $\mathbb{H}$.
(4)$\mathbb{H}/G$ Denotes quotient space.
(5) $S\mathbb{H}/G$ Denotes the unit tangent bundle over $\mathbb{H}/\Gamma$.
(6) The open set S in $\mathbb{H}/G$ is nice if the boundary $Fr(S) = adh(S) - S$ have finite 1-dimensional Hausdorff measure.
(7) $C_{00}^\infty(S\mathbb{H}/G)$ Denotes the set of "nice" test functions.
(8) γ_t Denotes the image of a segment in $\mathbb{H}/G$, this segment is obtained by projection of horocycle on the $\mathbb{H}/G$ focusing on the segment of h_t lying within the vertical strip $\{z, 0 \leq z \leq 1\}$. The length of γ_t is $\frac{1}{t}$. this means that: $\lim_{t\to 0} \gamma_t = +\infty$, and for any "nice" test open set like S in $\mathbb{H}/G$, we have:

$$\lim_{t\to 0} \frac{\text{length}(S \cap \gamma_t)}{\text{length}(\gamma_t)} = \frac{vol\ (S)}{vol(\mathbb{H}/G\ )}$$

(9) h_t denotes a horocycle in the upper half-plane of a constant imaginary part $y = t$.
(10)ν_t is the arc-length measure on the horocycle at height t.
(11)μ is the Poincare measure on $S\mathbb{H}/G$.
Proof: See Lagarias 2006 [37], Verjovsky 1993 [53].
Corollary 33: We have: $E[\ln(|\zeta(W)|] = 0$.
Recall that:

(1) E denotes expectation of Z^s as $E[Z^s] = \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$

(2)Gnedenko and Kolmogorov (see Biane and al. [6]) discovered that Riemann zeta function can be related to Brownian motion. Consider a Brownian motion on the line: $B_t \in \mathbb{R}, t \geq 0$, started at B_0 and with condition $B_1 =$

0.
Then:$Z = \max_{t\in[0,1]}(B_t) - \min_{t\in[0,1]}(B_t)$ denotes the length of the range of B_t.
(3) Considering a 2-dimensional Brownian motion in the plane, starting at (0, 0): W is such that the point $(\frac{1}{2}, W)$ is the first point of contact with the line $X = \frac{1}{2}$.
Proof: See Balazard-Saias-Yor 1999[3].
Corollary 34: (Salem integral equation) the integral equation $\int_0^{+\infty} \frac{e^{-\sigma-1}\phi(z)}{e^{\frac{x}{z}}+1} = 0$ has no non trivial bounded solution for $\frac{1}{2} < \sigma < 1$.
Proof: See Salem 1935 [50].
3) Because the RH implies the below properties, they are true:
Corollary 35: (Dudek) we have: $\forall x \geq 2 \exists p \in \mathbb{P}$ (prime integer) such that: $x - \frac{4}{\pi}\sqrt{x}\ln(x) \leq p \leq x$
Proof: see A. Dudek [19]
Corollary 36: (Cramer) $p_{k+1} - p_k = O(\sqrt{p_k}\ln(p_k))$
Proof: see M. Ghanim [27]
Corollary 37: we have: $\begin{cases} e^{\gamma} \leq \limsup_{t\to+\infty} \frac{\zeta(1+it)}{\ln(\ln(t))} \leq 2e^{\gamma} \\ \frac{6}{\pi^2}e^{\gamma} \leq \limsup_{t\to+\infty} \frac{1}{\zeta(1+it)\ln(\ln(t))} \leq \frac{12}{\pi^2}e^{\gamma} \end{cases}$
Proof: see Shashank Chorge [13], [14]
Corollary 38: the De Polignac and the twin primes conjectures are true.
Proof: see M. Ghanim [25].
Corollary 39: the Legendre and the Euler conjectures are true.
Proof: See M. Ghanim [26].
Corollary 40: The binary Goldbach conjecture is true.
Proof: see M. Ghanim [28].
Remark: for more details about the consequences of RH see Balazard [5], Broughan [11], Lachaud [34], Landau [38] and Sabihi-Dorostkar [49].

REFERENCES

[1] Alkantara, J-Bode (1993): An integral equation formulation of the Riemann hypothesis, Integral equations operator theory 17(2), pp 151-168.
[2] Alter, R. (1968): On a necessary condition for validity of the Riemann hypothesis for functions that generalize the Riemann zeta function, Trans. Amer. Math. Soc., 130, pp 55-74.
[3] Balazard, M.-Saias, E.- Yor, M. (1999): Notes sur la fonction ζ de Riemann II, Adv. Math. 123 (2), pp 284-287.
[4] Balazard, M.-Saias, E. (2000): The Nyman-Beurling, equivalent form for the Riemann hypothesis, Expo. Math. 18 (2), pp 131-138.
[5] Balazard, Michel (2010): un siècle et demi de recherches sur l'hypothèse de Riemann. SMF Gazette, 126, pp 7-24. Available at: https://smf.emath.fr/les-publications/la-gazette/ (Accessed on February 21, 2022).
[6] Biane, P.-Pitman, J.-Yor, M. (2001): Probability laws related to the Jacobi theta and Riemann zeta functions and Brownian excursions, Bull. Amer. Math. Soc. 38, pp 435-465.
[7] Bibmath: Math Spé. Exercises sur la Topologie des espaces vectoriels normés. Exercice 16: diamètre d'une partie bornée. Available at: https://www.bibmath.net/ressources/index.php?action=affiche&quoi=mathspe/feuillesexo/topoevn&type=fexo/ (Accessed on May 25, 2022).
[8] Bombieri, E.: Problems of the Millennium: the Riemann Hypothesis. Available at: http://www.claymath.org/millennium/Riemann_Hypothesis/official_Problem_description.pdf/ (Accessed on February 28, 2022).
[9] Bombieri, E-Lagarias, J. C (1999): Complements to Li's criterion for the Riemann hypothesis. J. Number Theory, 77(2), pp 274-287.
[10] Borwein, P-Choi, S-Rooney, B-Weirathmueller, A (2007): The Riemann Hypothesis, Springer.
[11] Broughan, Kevin (1917): equivalents of the Riemann Hypothesis. Vol1: arithmetic equivalents. Vol 2: Analytic equivalents. Published by the Cambridge University Press. The Vol 1 is Available at: https://www.cambridge.org/core/books/equivalents-of-the-riemann-hypothesis/6900C0A3B7B2ABBCE3EB285641F95A30/ The Vol 2 is Available at:

https://www.cambridge.org/core/books/equivalents-of-the-riemann-hypothesis/095D9AC2C5E0A74FF24389AD0D9BD458/ (Accessed on February 18, 2022).
[12] Calculateur: Logiciel donnant la décomposition des nombres entiers en facteurs premiers. Available at: https://fr.numere-prime.ro/comment-decomposer-nombre-en-produit-facteurs-premiers.php?nombre=50/ (Accessed on: February 21, 2022).
[13] Chorge, Shashank (October 2021): Extreme values of the Riemann zeta function at its critical points in the critical strip. Available at: http://arxiv.org/pdf/2110.14229.pdf/(Accessed on April 8, 2022).
[14] Chorge, Shashank (December 2021): Extreme values of the Dirichlet L-functions at critical points of the Riemann zeta function. Available at: http://arxiv.org/pdf/2110.14178.pdf/ (Accessed on April 8, 2022).
[15] Csordas, G. - Craven, T. (1989): Jensen Polynomials and the Turan and the Laguerre inequalities, Pacific J. Math .136, 2, pp 241-260.
[16] Csordas, G.- Varga, R .S (1990): Necessary and sufficient conditions and Riemann hypothesis, Adv. Appl. Math. 11, pp 328-357
[17] Csordas, G.- Odlyzko, A.-Smith, W.- Varga, S. (1993): A new Lehmer pair of zeros and a new lower bound for the Bruijn-Newman constant. Electron. Tans. Numer. Anal. 1, pp 104-111.
[18] Csordas, G.-Smith, A.-Varga, R.S. (1994): Lehmer pairs of zeros, the de Bruijn-Newman constant and the Riemann hypothesis, Constr. Approx. 10, pp 107-109.
[19] Dudek, Adrian (2014): On the Riemann Hypothesis and the difference between primes.Available at: http://arxiv.org/abs/1402.6417/ (Accessed on April 8, 2022).
[20] Edwards, H.M. (1974): Riemann zeta function. Academic Press (Reprinted by Dover publications 2001)
[21] Euclid (1966): Les éléments. T1 (les livres I-VII), T2 (les livres VIII-IX), T3 (les livres X-XIII) in les œuvres d'Euclide the French translation of the Euclid Grec work by F. Peyrard Blanchard. Paris. France. Available at: https://archive.org/details/lesoeuvresd'euclid03eucl/(Accessed on February 20, 2021)
[22] Ghanim, Mohammed (April 2017): confirmation of the Riemann Hypothesis. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS). India. April issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2017/April-2017/1.pdf/ (Accessed on February 18, 2021).
[23] Ghanim, Mohammed (May 2017): Confirmation of the irrationality of the Euler constant. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. May issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2017/May-2017/1.pdf/ (Accessed on February 18, 2022).
[24] Ghanim, Mohammed (May 2018): Confirmation of the Goldbach binary conjecture. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. May issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2018/May-2018/1.pdf/ (Accessed on February 18, 2022.
[25] Ghanim, Mohammed (July 2018): Confirmation of the De Polignac and the twin primes conjectures. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. July issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2018/July-2018/1.pdf/ (Accessed on February 18, 2022).
[26] Ghanim, Mohammed (August 2018): Confirmation of the Legendre and the Euler conjectures. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. August issue.Available at: http://www.gjaets.com/Issues%20PDF/Archive-2018/August-2018/1.pdf/ (Accessed on February 18, 2022).
[27] Ghanim, Mohammed (August 2020): Confirmation of the Firoozbakht conjecture and deduction of the Andrika, Cramer and Sinha conjectures with some other consequences. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. August issue.Available at: http://www.gjaets.com/Issues%20PDF/Archive-2020/August-2020/1.pdf/(Accessed on April 8, 2022).
[28] Ghanim, Mohammed (June 2021): Confirmation of the Goldbach binary conjecture by an elementary short proof. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. June issue.Available at: http://www.gjaets.com/Issues%20PDF/Archive-2021/June-2021/1.pdf/ (Accessed on February 18, 2022).
[29] Ghanim, Mohammed (December 2021): Confirmation of the Lemoine-Levy conjecture. Global Journal of Advanced Engineering Technologies & Sciences (GJAETS), India. December issue.Available at: http://www.gjaets.com/Issues%20PDF/Archive-2021/December-2021/1.pdf/ (Accessed on February 17, 2022).
[30] Gronwall, Thomas (1913): Some asymptotic expressions in the theory of numbers. Trans. Amer. Math.Soc, 14, PP113-122

[31] Guy, Richard K. (2004): Unsolved Problems in Number Theory New York: Springer-Verlag C1
[32] Hardy, G.H-Wright, E.M (1979): An Introduction to the theory of Numbers. Fifth edition. Oxford univ. Press. Oxford.
[33] Hardy, G.H-Littlewood, J.E (1918): Contributions to the theory of the Riemann zeta function and the theory of the distribution of the primes; Acta; Math, 41(1), pp 119-196.
[34] Lachaud, G (2001): l'hypothèse de Riemann. La recherche. Numero special 346.
[35] Lagarias, J.C. (1999): On a positivity property of the Riemann ξ–function, Acta. Arith. 89, 3, pp 217-234.
[36] Lagarias, J.C. (2002): An elementary Problem Equivalent to the Riemann Hypothesis. The American Mathematical Monthly. Vol 109, No 6, pp 534-543.
[37] Lagarias, J.C. (2006): The Riemann Hypothesis. Arithmetic and Geometry surveys. Noncom.Geom, 6, pp 127-141.
[38] Landau, E (1985): Collected works. Edited by L. Mirsky and al. Essen publisher. Thales Verlag.
[39] Li, X. J (1997): The positivity of a sequence of numbers and the Riemann hypothesis. J. Number Theory, 65 (2), pp 325-333.
[40] Liste des nombres premiers: Liste donnant les nombres premiers p avec leur ordre pour p≤ 10000.Available at: https://labri.fr/perso/betrema/deug/poly/annexes/premiers.html/(Accessed on: February 18, 2022).
[41] Massias, J.P.-Nicolas, J.L.-Robin, G. (1988): Evaluation asymptotique de l'ordre maximum d'un élément du groupe symétrique, Acta. Arith. 50(3), pp 221-242.
[42] Newman, C.M. (1976): Fourier transforms with only real zeros, Proc; Amer. Math. Soc. 61(2), pp 245-251.
[43] Odlyzko, A. (2000): An improved bound for the Bruijn-Newman constant, Numer. Algorithms. 25, pp 293-303.
[44] Polya, G. (1927): Uber die algebraisch-funktionentheoretischen untersuchungen von J.L.W.V.Jensen, Kgl. Danske Vid. Sel. Math. Fys. Medd. 7, pp 3-33.
[45] Ramanujan, Srinivasa (1915): Highly composite numbers. Proc. London Math. Soc. 14, pp 347-407.See also: S. Ramanujan: Collected papers, Chelsea 1962, pp 78-129, and notes pp 338-339.See also: S. Ramanujan, Highly composite numbers. Annotated and with a foreword by J.- L. Nicolas and G. Robin, Ramanujan J. 1 (1997) 119–153.
[46] Redheffer, R.M. (1977): Eine explicit losbare optimierungsaufgabe, Internat.Schritenreihe Num. Math. 142, pp 141-152.
[47] Riemann, Bernhard (1859): Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberchte der Berliner Akademie. November 1859.The Original German paper of Riemann is Available at: https (Accessed on April 8, 2022). The German Riemann paper, Transcribed by David, R. Wilkins on December 1998, is Available at: https://www.claymath.org/sites/default/files/zeta.pdf/(Accessed on April 8, 2022).
The French translation « sur le nombre des nombres premiers inférieurs à une taille donnée », of the Riemann paper, by Laugel is Available at: https://fr.wikisource.org/wiki/Sur_le_nombre_de_nombres_premiers_inf%C3%A9rieurs_%C3%A0_une_taille_donn%C3%A9e_(Riemann,_trad._Laugel)/(Accessed on April 4, 2022).
The English translation « On the Number of prime Numbers less than a given quantity » of the Riemann paper, by David R. Wilkins on December 1998, is Available at:https://www.claymath.org/sites/default/files/ezeta.pdf/(Accessed on April 7, 2022).
[48] Robin, Guy (1984): Grandes valeurs de la fonction Somme des diviseurs et hypothèse de Riemann. J.Math.Pures Appl. 63, pp 187-213.
[49] Sabihi, Ahmad - Dorostkar, Mohammed Ali (September 2021): Riemann Hypothesis equivalences, Robin Inequality, Lagarias criterion and Riemann Hypothesis. Available at: https://arxiv.org/abs/1605.08273/(Accessed on February 18, 2022).
[50] Salem, R. (1935): Sur une proposition équivalente à l'hypothèse de Riemann. C.R. Acad. Sci. Paris. 236, pp 1127-1128.
[51] Speiser, A (1934): Geometrisches Zur Riemannschen zetafunction, Math. Annalen, 110, pp 514-521.
[52] Titchmarsh, A. (1986): the theory of the Riemann zeta function, second edition, Clarendon press, Oxford University press, New York.
[53] Verjovsky, A. (1993): Arithmetic geometry and dynamics in the unit tangent bundle of the modular orbifold in: dynamical systems (Santiago 1990), Pitman. Res. Notes Math. N° 285, Longman .Sci. Tech. Harlow, pp 263-298.

[54] Verjovsky, A. (1994): Discrete measures and Riemann hypothesis. Kodai.Math.J. 17(3), pp 596-608.(See also workshop on Geometry and Topology –Hanoi 1993).
[55] Viossat, Y (2009): Quelques notions de logique. Cours DUM12E, Algèbre 1, 2009-2010, Université Paris-Dauphine. P 13.Available at:https://www.ceremade.dauphine.fr/~viossat/PDFs/algebre1/2009-10/logique-2009-10.pdf/(Accessed on February 18, 2022).
[56] Volchkov, V.V. (1995): On an equality equivalent to the Riemann hypothesis. Ukr. Math. Zh, 47 (3), pp 422-423.
[57] Wikipedia: The integer part. Available at: http://en.wikipedia.org/wiki/theinteger_part/(Accessed on February 18, 2022).
[58] Wikipedia: Topology. Available at: http://en.wikipedia.org/wiki/topology/(Accessed on February 18, 2022)
[59] Wikipedia: Adherence. Available at: http://en.wikipedia.org/wiki/the_adherence/(Accessed on February 18, 2022).
[60] Wikipedia: Continuity. Available at: http://en.wikipedia.org/wiki/continuity/(Accessed on February 18, 2022).
[61] Wikipedia: maximum and minimum. Available at: http://en.wikipedia.org/wiki/maximum_minimum/(Accessed on February 18, 2022).
[62] Wikipedia: Negation of a proposition. Available at: http://en.wikipedia.org/wiki/negation_of_a_proposition/(Accessed on February 18, 2022).
[63] Wikipedia: prime numbers. Available at: http://en.wikipedia.org/wiki/prime_numbers/(Accessed on February 18, 2022).
[64] Wikipedia: division and divisors s. Available at: http://en.wikipedia.org/wiki/division_divisors/(Accessed on February 18, 2022).
[65] Wikipedia: Partie entière et Partie fractionnaire. Available at : https://fr.wikipedia.org/wiki/Partie_enti%C3%A8re_et_partie_fractionnaire/ (Accessed on February 26, 2022).
[66] Wikipedia: Interior. Available at: https://en.wikipedia.org/wiki/interior/(Accessed on March 7, 2022).
[67] Wikipedia: Field. Available at: https://en.wikipedia.org/wiki/field/(Accessed on March 7, 2022).
[68] Wikipedia: Vector space. Available at: https://en.wikipedia.org/wiki/vectorial_space/(Accessed on March 7, 2022).
[69] Wikipedia: Normed vectorial space. Available at: https://en.wikipedia.org/wiki/normed_vectorial_space/(Accessed on March 7, 2022)
[70] Wikipedia: The boundary of a set. Available at: https://en.wikipedia.org/wiki/boundary/(Accessed on March 7, 2022).
[71] Wikipedia: The absolute value. Available at: https://en.wikipedia.org/wiki/absolute value/(Accessed on March 7, 2022).
[72] Wikipedia: Connected space. Available at: https://en.wikipedia.org/wiki/connected space/(Accessed on March 7, 2022).
[73] Wikipedia: Metric space. Available at: https://en.wikipedia.org/wiki/metric_space /(Accessed on March 7, 2022).
[74] Wikipedia: Groups. Available at: https://en.wikipedia.org/wiki/group/ (Accessed on October 14, 2022)
[75] Wikipedia: Rings. Available at: https://en.wikipedia.org/wiki/ring/(Accessed on October 14, 2022)
[76] Wikipedia: Ultra-metric absolute value. Available at: https://en.wikipedia.org/wiki/ultrametricabsolute value/(Accessed on October 14, 2022)
[77] Wikipedia: Hilbert space. Available at: https://en.wikipedia.org/Hilbert space/(Accessed on October 14, 2022)
[78] Wikipedia: A directed set. Available at: https://en.wikipedia.org/directedset/ (Accessed on October 14, 2022)
[79] Wikipedia: A partially ordered set. Available at: https://en.wikipedia.org/partiallyordredset/ (Accessed on October 14, 2022)
[80] Wikipedia: The inverse system of groups and homomorphisms. Available at: https://en.wikipedia.org/wiki/inversesystemofgroupsand_homomorphisms/(Accessed on October 14, 2022)
[81] Wikipedia: The space of p-adic numbers. Available at: https://en.wikipedia.org/wiki/spaceofpadicnumbers/(Accessed on October 2022)
[82] Wikipedia: Adelic rings. Available at: https://en.wikipedia.org/wiki/adelicrings/(Accessed on October 14, 2022)

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[83] Wilton, J.R. (1919): A note on the Ramanujan arithmetical function τ, Proc. Cambridge Philos. Soc. 25, pp 121-129.