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CONFIRMATION OF THE GOLDBACH BINARY CONJECTURE

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ABSTRACT

I show that the Goldbach (1690-1764) conjecture, remained open since 1742, is true by using some elementary tools of mathematics. The proof is essentially based on (the American Mathematician Lowell) Schoenfeld (1920-2002) inequality that I have proved in [5] and on the intermediate value theorem.

KEYWORDS: prime integer, prime-counting function, Goldbach conjecture, Schoenfeld inequality, intermediate value theorem 2010 Mathematics Subject Classification: A 11 xx (Number theory)

INTRODUCTION

Definition 1: We call « the Goldbach conjecture » or « the Goldbach strong conjecture » or « the Goldbach binary conjecture » or « the Goldbach problem » (according to D.Hilbert) or « the Goldbach theorem » (according to G.H.Hardy) the following assertion: "any even integer greater than 4 is the sum of two prime integers".

History: the Goldbach conjecture was announced by the German Mathematician Christian Goldbach (1690-1764) in a letter addressed to the Swiss Mathematician Leonhard Euler (1707-1783) on 7 June 1742 [6]. It has remained, without a rigorous poof from this date, although many attempts by the greatest mathematicians.

In 1900, the German Mathematician David Hilbert (1862-1943) said in his conference delivered before the second international congress of mathematicians hold at Paris in the 8th point about «the prime numbers problems »: « … and perhaps after an exhaustive discussion of the Riemann formula on prime numbers we will be in a position to reach the rigorous solution of the Goldbach problem i.e.: if any even integer is a sum of two positive prime integers? …» [8].

In 1940, the English Mathematician G.H.Hardy (1877-1947) writed: « it exists some theorems such 'the Goldbach theorem" which did not be proved and which any stupid could conjecture » [7] [13]

In 1977, the American Mathematician (of Polish descent) H.A.Pogorzelski (1922-2015) [11] affirmed to prove the Goldbach conjecture, but his proof is not generally accepted.

In 2000, Faber and Faber devoted $1000000 for any one proving the Goldbach conjecture between March 2000 and March 2002, but no one could give a proof and the question remained open [3][13].

However, the Goldbach conjecture was verified for all the entire even values of the integer $n, 4 \leq n \leq m$, where $m = 10^4$ by Desboves in 1885, $m = 10^5$ by N.Pipping in 1938, $m = 10^8$ by M.L.Stein and M.L.Stein in 1965, $m = 2.10^{10}$ by A.Granville and J.Van der lune and H.J.J Te Riele in 1989, $m = 4.10^{11}$ by M.K.Sinisalo in 1993, $m = 10^{14}$ by J.M.Deshouillers and H.J.J.Te Riele and Y.Saouter en 1998, $m = 4.10^{14}$ by J.Richstein in 2001, $m = 2.10^{16}$ by T.Oliveira E Silva on 3/14/2003 and $m = 6.10^{16}$ by T.Oliveira E Silva on 10/3/2003 (See[13] and its references)

Finally The Nice University (France) devoted online, since 1999, a site [17] giving, for higher values of chosen even natural numbers, all there Goldbach-decompositions in sums of two prime natural numbers.

The note: my purpose in the present brief note is to show that the Goldbach conjecture, remained open since 1742, is true by using some elementary tools of mathematical analysis. The proof is based essentially on the intermediate value theorem and the Schoenfeld inequality that I have proved in [5]. The main result of the present work is:

Theorem: $\forall n \in \mathbb{N}, n \geq 2, \exists (p, q)$ two prime numbers such that: $2n = p + q$.

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The paper is organized as follows. §1 is an introduction containing the necessary definitions and some history. The §2 gives the ingredients of the proof of our main result. The §3 gives the proof of our main result. §4 is the conclusion. The §5 gives the references of the paper for further reading.

INGREDIENTS OF THE PROOF:

Definition2: a natural integer p is said to be prime if its set of divisors is {1, p}.

Definition3: Define the set $\mathbb{P}$of prime integers by:
$\mathbb{P} = \{p \in \mathbb{N}, p$ is prime$\}= \{2=p_1,3=p_2,5=p_3,7=p_4,11=p_5,13=p_6,17=p_7,19=p_8,23=p_9, \ldots,p_k \ldots\}$.

Proposition1: (Euclid (3thd century before Jesus Christ)) (See [2]) $\mathbb{P}$ is an increasing infinite sequence$(p_k)_{k\in\mathbb{N}^*}$.
i.e.: $\forall k \in \mathbb{N}^*\ \ p_{k+1} > p_k$ i.e $\forall k \in \mathbb{N}^*\ p_{k+1} \geq p_k + 1$.

Proposition2: (some properties of the cardinality of a finite set) (See [16]) the cardinality of a finite set A, denoted card (A), is a natural number: the number of elements in the set A. We have the following properties:
(1)$card(\emptyset) = 0$ ($\emptyset$ is the empty set) and $card(\{a\}) = 1$
(2)$A \subset B \Rightarrow card(A) \leq card(B)$
(3)$card(A \cup B) = card(A) + card(B) - card(A \cap B)$
(4)$card(A\backslash B) = card(A) - card(A \cap B)$ with $A\backslash B = \{x, x \in A \ and \ x \notin B\}$ is the set difference of the sets A and B.
(5)$card(A \times B) = card(A)card(B)$ with $A \times B = \{(x,y), x \in A \ and \ y \in B\}$ is the Cartesian product of the sets A and B.
(6) $A_i \cap A_j = \emptyset$, for : $i \neq j$, $i,j = 1,2,\ldots,n \Rightarrow card(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} card(A_i)$

Definition4: Let, for $k \in \mathbb{N}$: $\mathbb{P}_k = \{p \in \mathbb{P}, p \leq k\}$ and $\pi(k) = card(\mathbb{P}_k)$ (which is the number of the elements in the finite set $\mathbb{P}_k$). The function $k \to \pi(k)$ is called the prime-counting function.

Definition5: We have:$f = O(g)$ in the neighborhood of $+\infty \Leftrightarrow \exists A > 0 \exists B \in \mathbb{R}$ such that:
$$t \geq B \Rightarrow |f(t)| \leq A|g(t)|$$

Proposition3: (The Schoenfeld inequality) (See [1], [5], [10], [12])
$$\forall n \geq 2657 \quad |\pi(n) - \int_0^n \frac{dt}{\ln(t)}| \leq \frac{\sqrt{n}\ln(n)}{8\pi}$$
Where :
* $\int_0^n \frac{dt}{\ln(t)} = \int_2^n \frac{dt}{\ln(t)} + R$
* $R = 1.04516378011749278484458888919413136522615578151 2\ldots$ (voir [14])
I.e. with the notation of definition 5 : $\pi(n) - \int_0^n \frac{dt}{\ln(t)} = O(\sqrt{n}\ln(n))$ in the neighborhood of $+\infty$.

Proposition4 : We have :
(i) $\lim_{n\to+\infty} a_n = s \in \mathbb{R} \Leftrightarrow \forall\epsilon > 0 \exists N \in \mathbb{N}$ such that$\forall n \geq N$: $|a_n - s| < \epsilon$
(ii) $\lim_{n\to+\infty} a_n = +\infty \Leftrightarrow \forall\epsilon > 0 \exists N \in \mathbb{N}$ such that$\forall n \geq N$: $a_n > \epsilon$
(iii)$\lim_{x\to a} f(x) = f(a) \Leftrightarrow \forall\epsilon > 0 \exists\delta > 0$: $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Proposition5: (See [4], p 31 and [9], p 4) Recall that for any sequences (x_n) and (y_n) of $\mathbb{R}$, we have :
(1)The number:$\text{limsup}(x_n) = inf_{n\in\mathbb{N}}(\sup(\{x_k; k \geq n\}) = \lim_{n\to+\infty} (\sup\{x_k, k \geq n\})$ exits always in $[-\infty, +\infty]$
(2)The number:$\text{liminf}\ (x_n) = sup_{n\in\mathbb{N}}(\inf(\{x_k, k \geq n\}) = \lim_{n\to+\infty} (\inf\{x_k, k \geq n\})$ exists always in $[-\infty, +\infty]$
(3)(i) $\text{limsup}(x_n) = -\text{liminf}\ (-x_n)$
(ii)$\text{liminf}x_n \leq \ \text{limsup}x_n$
(4) $\lim_{n\to+\infty} x_n = \text{limsup}(x_n) = \text{liminf}(x_n)$
(5) If y_n converges :
  (i) $\text{limsup}(x_n + y_n) = \text{limsup}(x_n) + \lim_{n\to+\infty} y_n$
  (ii) $\text{liminf}(x_n + y_n) = \text{liminf}(x_n) + \lim_{n\to+\infty} y_n$
(6) If (x_n) is bounded$\subset [a,b] \subset \mathbb{R}$, we have: $a \leq \text{liminf}x_n \leq \text{limsup}x_n \leq b$

(7)$(i)\text{limsup}x_n = s \in \mathbb{R} \Leftrightarrow \begin{cases} \forall\epsilon > 0 \exists n \in \mathbb{N} \forall k \geq n \;\; x_k < s + \epsilon \\ \forall\epsilon > 0 \forall n \in \mathbb{N} \exists k \geq n \;\; x_k > s - \epsilon \end{cases}$

(ii) $\text{liminf}x_n = s \in \mathbb{R} \Leftrightarrow \begin{cases} \forall\epsilon > 0 \exists n \in \mathbb{N} \forall k \geq n \;\; x_k > s - \epsilon \\ \forall\epsilon > 0 \forall n \in \mathbb{N} \exists k \geq n \;\; x_k < s + \epsilon \end{cases}$

Proposition6 : $\forall a, b, x, y \in \mathbb{R} : \begin{cases} a \leq b \\ 0 \leq x \leq y \end{cases} \Rightarrow ax \leq by$

Proposition7 if $y_n \geq 0 \;\; \forall n$, we have : $\text{liminf}\,(x_n y_n) \geq \text{liminf}(x_n)\,\text{liminf}(y_n)$

Proof : (of proposition7)
By proposition 6 and the definition of « the inf.», we have successively :

$\forall n \geq p: \begin{cases} 0 \leq inf_{n\geq p} y_n \leq y_n \\ inf_{n\geq p} x_n \leq x_n \end{cases} \Rightarrow \forall n \geq p: inf_{n\geq p} x_n inf_{n\geq p} y_n \leq x_n y_n \Rightarrow inf_{n\geq p} x_n inf_{n\geq p} y_n \leq inf_{n\geq p}(x_n y_n) \Rightarrow$
$\text{liminf}x_n \text{liminf}y_n = \lim_{p\to+\infty} inf_{n\geq p} x_n \lim_{p\to+\infty} inf_{n\geq p} y_n = \lim_{p\to+\infty} inf_{n\geq p} x_n inf_{n\geq p} y_n \leq \lim_{p\to+\infty} inf_{n\geq p}(x_n y_n) =$
$\text{liminf}(x_n y_n)$

Proposition8 : (i) Any non empty part $E$of $\mathbb{N}$ has a minimal element: $\min(E)$. $\min(E)$ is characterized by :$\min(E) \in E, \forall x \in E: x \geq \min(A)$, $\min(A) = 0$ or $\min(A) - 1 \notin E$
(ii)Any non empty part E of $\mathbb{N}$, bounded above, has a maximal element :$\max(E)$. $\max(E)$ is characterized by : $\max(E) \in E, \forall x \in E: x \leq \max(E)$ and $\max(E) + 1 \notin E$.

Proposition9 : if $I \subset \mathbb{R}$ is an interval, $a \in I$ and $f: I \to \mathbb{R}$ is a function, we have :
(1)f continuous in $a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$
(2) f continuous in $\alpha \Leftrightarrow \left(\left(\forall(\alpha_k) \text{ a sequence: } \lim_{k\to+\infty} \alpha_k = \alpha \right) \Rightarrow \left(\lim_{k\to+\infty} f(\alpha_k) = f(\alpha) \right) \right)$
(3)f continuous on $I \Leftrightarrow f$ continuous in any $a \in I$

Proposition10 : (Intermediate value theorem) (See [15]) Let $[a, b](a < b)$ an interval of $\mathbb{R}$ and f:$[a, b] \to \mathbb{R}$ a continuous function. If $f(a)f(b) < 0$, then :$\exists c \in]a, b[$ such that : $f(c) = 0$.

Proposition11 : Any sequence $(t_k) \subset [a, b]$ (an interval of $\mathbb{R}$), has a convergente subsequence, denoted also (t_k), with : $\lim_{k\to+\infty} t_k = t \in [a, b]$.

THE PROOF OF THE GOLDBACH BINARY CONJECTURE:

Theorem: $\forall n \geq 2 \exists (p, q) \in \mathbb{P}_{2n} \times \mathbb{P}_{2n}$ such that $2n = p + q$

Proof: (of the Theorem)
*Let:
$A_{2n} = \{(p, q) \in \mathbb{P}_{2n} \times \mathbb{P}_{2n}, p + q = 2n\}$ for an integer $n \geq 2$
$B_k = \{(p, q) \in \mathbb{P}_k \times \mathbb{P}_k, p + q \leq k\}$ for an integer $k \geq 4$.
*Show that: A_{2n} is $\neq \emptyset$.
*It is evident that the theorem is proved if we show that: $card(A_{2n}) > 0$.
*The proof of the theorem will be deduced from the lemmas below.

Lemma1 : We have :
(1) $card(B_{2n}) = \sum_{p\in\mathbb{P}_{2n}} \pi(2n - p)$
(2) $car(A_{2n}) = \sum_{p\in\mathbb{P}_{2n-1}} \pi(2n - p) - \pi(2n - p - 1)$

Proof: (of lemma1)
(1)*We have:$B_{2n} = \cup_{p\in\mathbb{P}_{2n}}(\{p\} \times \mathbb{P}_{2n-p})$ with: $\left(\{p\} \times \mathbb{P}_{2n-p}\right) \cap \left(\{q\} \times \mathbb{P}_{2n-q}\right) = \emptyset$ for: $p \neq q$.
*So, by proposition2, we have:

$$card(B_{2n}) = \sum_{p\in\mathbb{P}_{2n}} card\left(\{p\} \times \mathbb{P}_{2n-p}\right) = \sum_{p\in\mathbb{P}_{2n}} card\left(\mathbb{P}_{2n-p}\right) = \sum_{p\in\mathbb{P}_{2n}} \pi(2n - p)$$

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(2)* Noting that $2n$ being not prime, for $n \geq 2$, we have: $\mathbb{P}_{2n} = \mathbb{P}_{2n-1}$
*But: $B_{2n} = A_{2n} \cup B_{2n-1}$ with: $A_{2n} \cap B_{2n-1}=\emptyset$.
*So :$card(A_{2n})=card(B_{2n}) - card(B_{2n-1}) = \sum_{p\in\mathbb{P}_{2n}} \pi(2n-p) - \sum_{p\in\mathbb{P}_{2n-1}} \pi(2n-1-p)$
$=\sum_{p\in\mathbb{P}_{2n-1}} \pi(2n-p) - \pi(2n-p-1)$

Lemma2: We have: $\forall n \geq 2658$
$-\frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2) \leq 0 \leq \sum_{p\in\mathbb{P}_{2n-1}\setminus\mathbb{P}_{2n-2658}}\left(\pi(2n-p)-\pi(2n-p-1)\right) \leq \frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))$

Proof: (of lemma2)
*We have:
$\sum_{p\in\mathbb{P}_{2n-1}\setminus\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) \leq \sum_{k=2n-2657}^{2n-1}\left(\pi(2n-k)-\pi(2n-k-1)\right) = \sum_{k=1}^{2657}\pi(k) - \pi(k-1) = \pi(2657)-\pi(0) =\pi(2657)$
*We have:
$$n \geq 2658 \Rightarrow 0 \leq \sum_{p\in\mathbb{P}_{2n-1}\setminus\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) \leq \pi(2657) = \pi(2758) = 384 \leq \frac{1}{4\pi}\pi(2658)\sqrt{5314}\ln(5314) = 19108.3526 < \frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)$$
*So:
$-\frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2) \leq \sum_{p\in\mathbb{P}_{2n-1}\setminus\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) \leq \frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))$

Lemma3: We have, $\forall\, n \geq 2658$:
$$card(A_{2n}) = \sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1 \frac{dt}{\ln(t+2n-p-1)} + O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))$$
With:$\forall n \geq 2658$ $\left|\frac{O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\right| \leq \frac{1}{2\pi}$

Proof: (of lemma3)
*$2n-2658 \geq 2 \Leftrightarrow 2n \geq 2660 \Leftrightarrow n \geq 1330$
*By the assertion (2) of lemma1, we have:
$card(A_{2n}) = \sum_{p\in\mathbb{P}_{2n-1}}(\pi(2n-p)-\pi(2n-p-1)) = \sum_{p\in\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) + \sum_{p\in\mathbb{P}_{2n-1}\setminus\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1))$
*By the Schoenfeld inequality (See proposition 3), we have, for any prime integer p such that $2n-p-1 \geq 2657 \Leftrightarrow p \leq 2n-2658 \Leftrightarrow p \in \mathbb{P}_{2n-2658}$:
$$\begin{cases} \int_0^{2n-p}\frac{dt}{\ln(t)} - \frac{\sqrt{2n-p}\ln(2n-p)}{8\pi} \leq \pi(2n-p) \leq \int_0^{2n-p}\frac{dt}{\ln(t)} + \frac{\sqrt{2n-p}\ln(2n-p)}{8\pi} \\ -\int_0^{2n-p-1}\frac{dt}{\ln(t)} - \frac{\sqrt{2n-p-1}\ln(2n-p-1)}{8\pi} \leq -\pi(2n-p-1) \leq -\int_0^{2n-p-1}\frac{dt}{\ln(t)} + \frac{\sqrt{2n-p-1}\ln(2n-p-1)}{8\pi} \end{cases}$$
$$\Rightarrow \int_{2n-p-1}^{2n-p}\frac{dt}{\ln(t)} - \frac{\sqrt{2n-p}\ln(2n-p)}{4\pi} \leq \pi(2n-p)-\pi(2n-p-1) \leq \int_{2n-p-1}^{2n-p}\frac{dt}{\ln(t)} + \frac{\sqrt{2n-p}\ln(2n-p)}{4\pi}$$
$$\Rightarrow \sum_{p\in\mathbb{P}_{2n-2658}}\int_{2n-p-1}^{2n-p}\frac{dt}{\ln(t)} - \frac{1}{4\pi}\sum_{p\in\mathbb{P}_{2n-2658}}\sqrt{2n-p}\ln(2n-p) \leq \sum_{p\in\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) \leq \sum_{p\in\mathbb{P}_{2n-2658}}\int_{2n-p-1}^{2n-p}\frac{dt}{\ln(t)} + \frac{1}{4\pi}\sum_{p\in\mathbb{P}_{2n-2658}}\sqrt{2n-p}\ln(2n-p)$$
*Then, by the variable change $t = u+2n-p-1$ in the involved integals and by noting that:$\sum_{p\in\mathbb{P}_{2n-2658}}\sqrt{2n-p}\ln(2n-p) \leq card(\mathbb{P}_{2n-2658})\sqrt{2n-2}\ln(2n-2) = \pi(2n-2658)\sqrt{2n-2}\ln(2n-2)$, we have:
$$-\frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2) \leq \sum_{p\in\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)) - \sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)} \leq \frac{1}{4\pi}\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))$$
*So, by the assertion (2) of lemma1 and by lemma2, we have successively, for $n \geq 2658$:

$\left(\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}-\frac{1}{2\pi}\left(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)\right)\right)\leq card(A_{2n})=\sum_{p\in\mathbb{P}_{2n-1}}(\pi(2n-p)-\pi(2n-p-1))=\sum_{p\in\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1))+\sum_{p\in\mathbb{P}_{2n-1}\backslash\mathbb{P}_{2n-2658}}(\pi(2n-p)-\pi(2n-p-1)\leq\left(\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}+\frac{1}{2\pi}\left(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)\right)\right)$

*Finally, with the notation of definition3, we have:

$card(A_{2n})=\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}+O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))$

Lemma4: We have: (1)$0<\frac{\pi(2n-2658)}{\ln(2n-2)}\leq\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}\leq\frac{\pi(2n-2658)}{\ln(2657)}$

(2) $\lim_{n\to+\infty}\frac{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}=0(\Leftrightarrow\lim_{n\to+\infty}\frac{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}=+\infty)$

Proof: (of lemma4)

(1)$\forall n\geq 2658\ \forall t\in[0,1]\forall p\in\mathbb{P}_{2n-2658}$, we have:

$\ln(2657)\leq\ln(t+2n-p-1)\leq\ln(2n-2)$

$\Leftrightarrow\frac{1}{\ln(2n-2)}\leq\frac{1}{\ln(t+2n-p-1)}\leq\frac{1}{\ln(2657)}\Rightarrow 0<\frac{1}{\ln(2n-2)}\leq\int_0^1\frac{dt}{\ln(t+2n-p-1)}\leq\frac{1}{\ln(2657)}$

$\Rightarrow 0<\frac{\pi(2n-2658)}{\ln(2n-2)}\leq\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}\leq\frac{\pi(2n-2658)}{\ln(2657)}$

(2)*So:$0<\frac{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\leq\frac{1}{\sqrt{2n-2}\ln(2n-2)\ln(2657)}$

And $\lim_{n\to+\infty}\frac{1}{\sqrt{2n-2}\ln(2n-2)\ln(2657)}=0\Rightarrow\lim_{n\to+\infty}\frac{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}=0$

Lemma5: We have:

$0\leq\liminf\frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}=\liminf\frac{O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\leq\frac{1}{2\pi}$

Proof: (of lemma5)

*The inequalities follow by use of the assertion (6) of proposition 5, because: $\frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\geq 0$ and $\frac{O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\leq\frac{1}{2\pi}\quad\forall n\geq 2658$

*The equalities follow by use of lemma3, the relations (i), (ii) of the assertion (5) of proposition 5 and the relation (2) of lemma4.

Lemma6: $\exists N\geq 2658$ such that:$\forall n\geq N\ card(A_{2n})>0$

Proof : (of lemma6)

First case : if $\liminf\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}=+\infty$

*We have, by the assertion (4) of proposition5 : $\lim_{n\to+\infty}\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}=+\infty$

*So, by the assertion (ii) of proposition 3 :$\exists N\geq 2658$ such that $\forall n\geq N\ card(A_{2n})>0$

Second case : if $\liminf\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}=s<+\infty$

*By the assertion 5(ii) of proposition 5, the assertion (8) of proposition5, lemma3 and lemma5, we have successively:

$s-1=\liminf\left(\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)-1=\liminf\left(\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}-1\right)$

$=\liminf\left(\frac{O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)=\liminf\left(\frac{O(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2))}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\times\frac{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)$

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$$\geq \liminf\left(\frac{O\left(\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)\right)}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\right)\liminf\left(\frac{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)$$

$$=\liminf\left(\frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\right)\liminf\left(\frac{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)\geq 0$$

Remark: We have, in this case, necessarily: $\liminf\frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}=0$, because: $\liminf\left(\frac{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}\right)=+\infty \Rightarrow$ (if: $\liminf\frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\neq 0$), that : $\liminf\frac{card(A_{2n})}{\sum_{p\in\mathbb{P}_{2n-2658}}\int_0^1\frac{dt}{\ln(t+2n-p-1)}}=+\infty$.

*That is : $1\leq s<+\infty$

*So, by the assertion 7(ii) of proposition 5 (written for : $\epsilon=\frac{1}{2}$), we have :

$$\exists N\geq 2658\ \forall n\geq N:\ \frac{card(A_{2n})}{\pi(2n-2658)\sqrt{2n-2}\ln(2n-2)}\geq s-\frac{1}{2}\geq 1-\frac{1}{2}=\frac{1}{2}>0$$

*That is : $\exists N\geq 2658\ \forall n\geq N$ such that : $card(A_{2n})>0$.

Lemma7 : $\forall n\in\{2,3,4,5,\ldots,2657\}\ card(A_{2n})>0$ and $card(A_0)=card(A_2)=0$

Proof : (of lemma7)
See Wims [17] which is an online site devoted, by the Nice University (France), to giving, for very heigh values of even positive integers, all there Goldbach-decompositions in two prime positive integers.

Lemma8 : $\forall n\geq 2\ \ card(A_{2n})>0$.

Proof : (of lemma8)
Consider the subsets A,B and C of $\mathbb{N}$ defined by :
$A=\{N\in\mathbb{N},\forall n\geq N\ \ card(A_{2n})>0\}$
$B=\{N\in\mathbb{N},\exists n\geq N\ card(A_{2n})=0\}$
$C=\{N\in\mathbb{N},card\left(A_{2(N)}\right)=0\}$

Claim1 : The set A has a minimal element : $\min(A)=m\geq 2$

Proof : (of claim1)
*By lemma6, the subset $A=\{N\in\mathbb{N},\forall n\geq N\ \ card(A_{2n})>0\}$ of $\mathbb{N}$ is $\neq\emptyset$ (non empty).
*So, by proposition 8(i), A has a minimal element $m=\min(A)$.
*By lemma7 : $card(A_0)=card(A_2)=0\Rightarrow m\geq 2$

Claim2 : $B=\mathbb{N}\backslash A$

Proof : (of claim2)
*We have : $N\in\mathbb{N}\backslash A\Leftrightarrow N\notin A\Leftrightarrow\exists n\geq N$ such that : $card(A_{2n})=0\Leftrightarrow N\in B$
*So the result follows.

Claim3 : B is finite with $\max(B)=m-1$

Proof : (of lemma3)
*By claim1 :$\min(A)=m\geq 2$.
*So, because : $m-1\notin A$, we have : $card\left(A_{2(m-1)}\right)=0$ and : $m-1\in B$.
*Because : $m\in A$, we have : $\forall n\geq m\ \ card(A_{2n})>0$.
*Suppose that $\exists N\in B$ such that : $N>m-1$ i .e. $N\geq m$
*By definition of N :$\exists p\geq N$ such that : $card\left(A_{2p}\right)=0$
* But : $m\in A\Rightarrow\forall n\geq m\ \ card(A_{2n})>0$.
*So : $p\geq N\geq m\Rightarrow 0<card\left(A_{2p}\right)=0$

*This being contradictory, we have : $\forall N \in B\ \ N \leq m-1$
*The result follows

Claim4 : (i)$\forall N \in \mathbb{N} : N \in A \Rightarrow N+1 \in A$
(ii)So :$A = \{m, m+1, m+2, \dots\} = \{N \in \mathbb{N}, N \geq m\}$

Proof : (of claim4)
(i)Let $N \in A$, we have : $\forall n \geq N: card(A_{2n}) > 0 \Rightarrow \forall n \geq N+1 \geq N: card(A_{2n}) > 0$ i.e. $N+1 \in A$.
(ii)So : $m = \min(A) \Rightarrow A = \{m, m+1, m+2, \dots\}$

Claim5 : $B = \{0,1,\dots,m-1\} = \{N \in \mathbb{N}, 0 \leq N \leq m-1\}$

Proof : (of claim5)
The result is obtained by combination of claim2 and the assertion (ii) of claim4.

Claim6 : We have :
$B = \{0,1,\dots,m-1\} = \{N \in \mathbb{N}, \exists n \geq N\ card(A_{2n}) = 0\} \supset C = \{N \in \mathbb{N}, card\left(A_{2(N)}\right) = 0\}$

Proof : (of claim6)
*If $N \in C$, we have :$card(A_{2N}) = 0$. So :$\exists n = N \geq N$ such that :$card(A_{2n}) = 0$.
*That is :$C \subset B$

Claim7 : We have $\max(C) = m-1$

Proof : (of lemma7)
*We have :$card\left(A_{2(m-1)}\right) = 0 \Rightarrow m-1 \in C$.
*We have : $card(A_{2m}) > 0 \Rightarrow m \notin C$.
*We have : $C \subset B = \{0,1,\dots,m-1\} \Rightarrow \forall N \in C: N \leq m-1$.
*This shows that : $\max(C) = m-1$

Claim8 : we have $m = 2$

Proof : (of lemma8)
*We have :$m \geq 2$. suppose contrarily that $m \geq 3$.
*The claim 8 will be deduced from the under-claims below.

Under-claim1 : We have :
$m-1 \in C \Leftrightarrow \forall p,q \in \mathbb{P}_{2(m-1)} |2(m-1)-p-q| > 0$

Proof : (of the under-claim1)
The result follows by definition of $C = \{N \in \mathbb{N}, card(A_{2N}) = 0\}$ and by definition of $A_{2N} = \{(p,q) \in \mathbb{P}_{2N} \times \mathbb{P}_{2N}, 2N = p+q\}$

Under-claim2 : We have
(i)Let: n is a positive integer $\leq m-1$, $a,b \in \mathbb{P}_{2n}, p,q \in \mathbb{P}_{2(m-1)}$ and $k,r \in \mathbb{N}^*$.
(1) if : $|2n-a-b| \geq |2(m-1)-p-q|$, we have : $|2n-a-b| > 0$
(2) if : $|2n-a-b| < |2(m-1)-p-q|$, then :
(a)$\exists \theta(n,k,r,a,b,p,q) = \alpha_{k,r} \in]\frac{1}{2}, \frac{2}{3} - \frac{1}{r}[$ suchthat :

$$\alpha_{k,r}(|2n-a-b| + \tfrac{1}{k})^{\frac{2}{3}-\alpha_{k,r}} = (1-\alpha_{k,r})(|2(m-1)-p-q| + \tfrac{1}{k})^{\frac{2}{3}-\alpha_{k,r}}$$

(b)$\exists \alpha = \alpha_r \in]\frac{1}{2}, \frac{2}{3}[$ such that :

$$\alpha(|2n-a-b|)^{\frac{2}{3}-\alpha} = (1-\alpha)(|2(m-1)-p-q|)^{\frac{2}{3}-\alpha}$$

(ii)$\forall a,b \in \mathbb{P}_{2n}\ |2n-a-b| > 0$
(iii)$\forall 0 \leq n \leq m-1 : card(A_{2n}) = 0$

(iv)$C = B$

Proof : (of the under-claim2)
(i) (1) the result follows by the under-claim1.
(2)(a) Consider, on $[\frac{1}{2}, \frac{2}{3} - \frac{1}{r}]$, the continuous function $f_{k,r}$ defined by :

$$f_{k,r}(t) = t(|2n - a - b| + \frac{1}{k})^{\frac{2}{3}-t} - (1-t)(|2(m-1) - p - q| + \frac{1}{k})^{\frac{2}{3}-t}$$

*By the hypothesis «$|2n - a - b| < |2(m-1) - p - q|$ », we have :
** $f_{k,r}\left(\frac{1}{2}\right) = \frac{1}{2}((|2n - a - b| + \frac{1}{k})^{\frac{1}{6}} - (|2(m-1) - p - q| + \frac{1}{k})^{\frac{1}{6}}) < 0$
**$f_{k,r}\left(\frac{2}{3} - \frac{1}{r}\right) = (\frac{2}{3} - \frac{1}{r})(|2n - a - b| + \frac{1}{k})^{\frac{1}{r}} - \left(\frac{1}{3} + \frac{1}{r}\right)(|2(m-1) - p - q| + \frac{1}{k})^{\frac{1}{r}} > 0$, for a great positive integer r, because, by the assertion (i) of proposition 4 (written for : $\epsilon = \frac{1}{6}$), we have :
$\lim_{r\to+\infty} f_{k,r}(\frac{2}{3} - \frac{1}{r}) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \Rightarrow \exists N \in \mathbb{N}\ \forall r \geq N: f_{k,r}\left(\frac{2}{3} - \frac{1}{r}\right) \geq \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$
*So, by the intermediate value theorem (See proposition 10), we have :

$$\exists \alpha_{k,r} \in]\frac{1}{2}, \frac{2}{3} - \frac{1}{r}[\text{ such that } f_{k,r}(\alpha_{k,r}) = 0$$

*The result follows.
(b) and (ii) *By proposition 11, the bounded sequence$(\alpha_{k,r})_k \subset]\frac{1}{2}, \frac{2}{3} - \frac{1}{r}[$ (for a fixed r and a variable k), has a convergente subsequence, denoted also$(\alpha_{k,r})_k$, such that : $\lim_{k\to+\infty} \alpha_{k,r} = \alpha_r = \alpha \in [\frac{1}{2}, \frac{2}{3} - \frac{1}{r}]$.
* By proposition 9 : $\lim_{k\to+\infty} f_{k,r}(\alpha_{k,r}) = 0 \Leftrightarrow \alpha(|2n - a - b|)^{\frac{2}{3}-\alpha} = (1-\alpha)(|2(m-1) - p - q|)^{\frac{2}{3}-\alpha} > 0 \Leftrightarrow card(A_{2n}) = 0 \Leftrightarrow n \in C$

Remark : $\alpha \neq \frac{1}{2}$, because of our hypothesis «$|2n - a - b| < |2(m-1) - p - q|$ »
(iii) The result follows by combination of (i), (ii) of the under-claim2.
(iv)The result follows by combination of claim6, claim7 and the assertion (iii) of the under-claim 2.

Under-claim3 : We have :$m = 2$

Proof : (of under-claim3)
*By lemma 7, we have : $2 \notin C$.
*So, by the assertion (iv) of the under-claim 2 : $m - 1 = \max(C) < 2 \Leftrightarrow m < 3$.
*This contradicting our hypothesis « $m \geq 3$ », we have effectively showed that : $m = 2$.

CONCLUSION
I have showed the Goldbach binary conjecture via the Schoenfeld inequality, using elementary mathematical analysis tools, notably the intermediate value theorem.

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