A VARIANT OF THE PROOF OF THE FERMAT LAST THEOREM

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ABSTRACT
In the 13 pages paper [7] published by the GJAETS, I have given a short elementary proof of the Fermat's last theorem based essentially on the intermediate value theorem, the B. Bolzano (1781-1848)-K. Weierstrass (1815-1897) theorem and the L. Euler (1707-1783)-J.C.F. Gauss (1777-1855) theorem . I have deduced the Fermat last theorem in my paper [8] resolving affirmatively the Beal conjecture and in my paper [9] resolving affirmatively the abc-conjecture. The two papers being published by the GJAETS in February 2021 and February 2022 respectively. In the present paper I give also another shorter elementary proof of this conjecture based only on the intermediate value theorem and the increasing properties of some elementary functions, using some techniques of my previous papers [5], [6], [8] and [9]. Recall that the hard problem of finding an elementary proof of this famous conjecture (called the Fermat last theorem), saying that the Diophantine equation $x^n + y^n = z^n$ has, for $n \geq 3$, no natural integer solutions x, y, z such that: $0 < x < y < z$, remained open since 1665(the extinction date of the French Mathematician Pierre de Fermat (1601-1665)). Recall also that the 1994 proof of the English Mathematician Andrew Wiles (Born in1953) [23] is not elementary because it uses powerful tools of number theory and is not short because it takes 100 pages.

KEYWORDS: Fermat last theorem, Intermediate value theorem, increasing function, decreasing function.
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INTRODUCTION

Definition1: We call « Fermat last theorem », the following statement: « It does not exist natural integers x, y and z such that: $0 < x < y < z$ and $x^n+y^n=z^n$, for n a natural integer ≥ 3 ».

History: *This problem has appeared about the fourth century with the Greek mathematician Diophante (325-410) in his work « Arithmetica » [1] (the problem II.VIII, page 85), but the problem $x^2 + y^2 = z^2$, has appeared and was resolved by Euclid, about 300 before J.C, in his famous "Elements" (The Book X)[2]

*About 1621, the French mathematician Pierre Simon de Fermat (1601-1665) writed in the margin of the page 85 of his copy of [1] nears the statement of the famous problem, the following: « J'ai trouvé une merveilleuse démonstration de cette proposition, mais la marge est trop étroite pour la contenir ». This can be translated as:"I have discovered a truly remarkable proof which this margin is too small to contain". But it seems that Fermat has never published his proof. In any case we don't know now this proof.

*In 1670, the proof of the case n=4, by Fermat, was published by his son Samuel.

*On 4 August 1753, L. Euler wrote to Goldbach claiming to prove the Fermat last theorem for n=3, but his proof, published in his book "Algebra" (1770) is incomplete (see [3], [10]).

*In 1816, the Paris Sciences Academy devoted a gold medal and 3000 F for who can give a proof of the Fermat last theorem. This offer was retaken in 1850.

*In 1825, Dirichlet (1805-1859) and Legendre (1752-1833) proved the case n=5.

*Searching a solution to the Fermat last theorem, Marie-Sophie Germain (1776-1831) discovered his "Sophie Germain theorem"[13] which says that at least one of the positive integers x, y, z such that $x^p + y^p = z^p$,for $p \geq 3$ a prime integer, must be divisible by p^2 if we can find an auxiliary prime q such that:
1) Two none zero consecutive classes modulo q cannot be simultaneously be p-powers
2) p itself cannot be a p-power modulo q
*In 1832, Dirichlet proved the case n=14.
*In 1839, Lame proved the case n=7
*In 1863, the proof of the case n=3, by Gauss, was published.

*In 1908, The Gottingen University and the Wolfskehl Foundation devoted a price of 100.000 Marks for who can give a proof of the Fermat last theorem before 2008.

*In 1952, Harry Vandiver used a Swac Computer to show that the Fermat last theorem is true for $n \leq 2000$

*Between 1964 and 1994, Jean-Pierre Sere, Yves Hellegouarch and Robert Langlands have given some development to the problem by working on the representation of the elliptic curves with the modular functions.

* This problem, then, remained open for more than 370 years, (Although many attempts of the more eminent mathematicians), when in 1994 the English mathematician Andrew Wiles [23] proved it–by a relatively long proof that has occupied about 100 pages- using powerful tools of number theory, such the Shimura-Taniyama-Weil conjecture, the modular forms, the Galoisian representations…

So the problem of finding a short elementary proof of the Fermat last theorem remained open up to now.
*For More and detailed History see on Wikipedia the articles on the Fermat last theorem specially [12] with their references and the Simon Singh good book "Le dernier théorème de Fermat" [11].

The note: The present short note gives a variant of proof of the Fermat's last theorem different of that given in my precedent work [7] published by the GJAETS in December 2018. The proof is based only on the intermediate value theorem and the increasing properties of some elementary functions .

Results: our main result is

Theorem: (resolving the Fermat last theorem) let $n \geq 2$ a natural integer.

$$\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^n + b^n = c^n \text{ and } 0 < a < b < c \Rightarrow n = 2$$

Methods: I show that: we can suppose $a \geq 3$. By the Intermediate value theorem I show that:

$$\forall k \geq 2 \exists \lambda(k) \in]1 - \frac{\pi}{4}, 1[\text{ Such that: } \left(\frac{a}{c}\right)^{n\lambda(k)} \left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} + \left(\frac{b}{c}\right)^{n\lambda(k)} \left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} = 1$$

Then I show that: $\exists p \geq 2$ such that: $2 \leq n \leq \frac{2\tan(1-\lambda(p))}{1-\lambda(p)} \leq 2\left(\frac{4}{\pi}\right) = \frac{8}{\pi} = 2.546..$

So: $n = 2$

Organization of the paper: The paper is organized as follows. The §1 is an introduction giving the necessary definition and some history. The § 2 gives the ingredients of the proof. The §3 gives the variant of the proof of the Fermat last theorem. The §4 gives the references of the paper for further reading.

THE INGREDIENTS OF THE PROOF OF FERMAT LAST THEOREM

We will need the following results:
Definition2: (division and divisors [16]) let $b \neq 0, c$ two integers. We say that "b divides c" if $\exists a$ an integer such that: $c = ab$. We note $D(c)$ the set of divisors of the integer c.

Definition3: (prime integer [17]) a positive integer p is prime if $D(p) = \{1, p\}$. For example 2 is prime, it is the smallest prime integer. It is the single even prime integer.

Proposition1: (The Gauss theorem [20]) if p is a prime integer, then p divides the integer $c^m \Rightarrow p$ divides the integer c.

Proposition2: (Euclid (300 before J.C) theorem ([2], book X)) we have:
*for $n = 1$: the equation x+y=z has an infinite number of solutions.

*for $n = 2$: (x, y, z)= (3, 4, 5) is a solution of $x^2 + y^2 = z^2$, and we can show easily that: $\begin{cases} x = 2abc \\ y = a(b^2 - c^2) \\ z = a(b^2 + c^2) \end{cases}$, with $a, b, c \in \mathbb{N}$, three natural integers of different parity such that: $b > c$, are the solutions of: $x^2 + y^2 = z^2$

Proposition 3 :(Intermediate value theorem [15]) let $\varphi: [a, b] \to \mathbb{R}$ (With: $a < b$) a continuous function. Then:

$$\varphi(a)\varphi(b) < 0 \Rightarrow \exists c \in]a, b[\text{ such that } \varphi(c) = 0$$

Proposition 4: (The Newton binomial [21]) we have: $(a+b)^m = \sum_{k=0}^{m} C_m^k a^k b^{m-k}$ with C_m^k = the binomial coefficients=$\frac{m!}{k!(m-k)!}$, $k! = \prod_{i=1}^{k} i$ and $0! = 1! = 1$

Proposition5: (see [22])$\forall m \in \mathbb{N}^* \forall a, b \in \mathbb{R}$: $a^m - b^m = (a-b)\sum_{k=0}^{m-1} a^k b^{m-1-k}$

Proposition 6: (The Weierstrass-Bolzano theorem [14]) any bounded real sequence $(x_n)_n \subset [\alpha, \beta]$ has a subsequence, denoted for convenience $(x_n)_n$ also, converging to $x \in [\alpha, \beta]$

Proposition 7: (circular functions [19]) we have:
(0)$\tan(0) = 0$ and $\tan\left(\frac{\pi}{4}\right) = 1$ with $\tan(t) = \frac{\sin(t)}{\cos(t)}$
(1)$\forall t \in [0, \frac{\pi}{2}]$ $1 \geq \cos(t) \geq 0$ And $1 \geq \sin(t) \geq 0$
(3)The function $t \to \sin(t)$ is increasing on $[0, \frac{\pi}{2}]$ with$(\sin(t))' = \cos(t)$.
(4) The function $t \to \cos(t)$ is decreasing on $[0, \frac{\pi}{2}]$ with$(\cos(t))' = -\sin(t)$.
(5)The function $t \to \tan(t)$ is derivable on $]0, \frac{\pi}{2}[$ with $(\tan(t))' = \left(1 + (\tan(t))^2\right) = \frac{1}{(\cos(t))^2} > 0$ and has a reciprocal function denoted "arctan": $[0, +\infty[\to [0, \frac{\pi}{2}[$

Proposition 8: (The l'Hôpital rule [18])
(i) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$ (a can be infinite) the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called to be an indeterminate form (IF)$\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively.
(ii) If f, g are differentiable on an interval$]a, b[$ except perhaps in a point $c \in]a, b[$, if $\lim_{x\to c} \frac{f(x)}{g(x)}$ is the IF $\frac{0}{0}$ and if $\forall x \neq c, g'(x) \neq 0$, then: $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ when the limits have a sense.
(iii) If f', g' satisfies the same conditions as f and g the process is repeated.
(iv)The result remain true in the case where $\lim_{x\to c} \frac{f(x)}{g(x)}$ is the IF $\frac{\infty}{\infty}$.

THE VARIANT OF THE POROOF OF THE FERMAT LAST THEOREM

Theorem: (Proving the Fermat last theorem) let n an integer≥ 2 . Then:

$$\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^n + b^n = c^n \Rightarrow n = 2$$

Proof :(of the theorem)
*Let n an integer ≥ 2, suppose that $\exists\ a, b, c \in \mathbb{N}^*$ such that: $a^n + b^n = c^n$ and show that $n = 2$
*The proof of the theorem will be deduced from the below lemmas.

Lemma1: we can suppose $0 < a < b < c$

Proof: (of lemma1)
*We have:$a^n = c^n - b^n > 0 \Rightarrow b^n < c^n \Rightarrow b < c$.
*The order " $\leq$ " being total on $\mathbb{N}$, we have:$b \geq a$ *or* $a \geq b$
*So, we can suppose: $a \leq b$
*But: $n \geq 2 \Rightarrow a \neq b$. Indeed, if not we have:$a^n + b^n = 2b^n = c^n$ and $c \neq 0 \Rightarrow$ (by the Gauss theorem, 2 being a prime integer): 2 is a divisor of c. writing $c = 2^p q$ with q an odd integer, we have:$b^n = 2^{pn-1} q^n$. Then b=$2^r l$ with l an odd integer, so: $2^{rn} l^n = 2^{pn-1} q^n$.
*l, q being odd we have necessarily $rn = pn - 1 \Leftrightarrow n(p - r) = 1$
*So: $n = 1$.
*This contradicting our hypothesis:"$n \geq 2$", we have well: $a \neq b$
*So, we can suppose: $a < b$

Lemma2: we have: $c < a + b$

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Proof: (of lemma2)
*By the Newton binomial, and the hypothesis "$a^n + b^n = c^n$", we have: $(a+b)^n = a^n + b^n + \sum_{k=1}^{n-1} C_n^k a^k b^{n-k} = c^n + \sum_{k=1}^{n-1} C_n^k a^k b^{n-k}$
*So: $\sum_{k=1}^{n-1} C_n^k a^k b^{n-k} > 0 \Rightarrow (a+b)^n > c^n \Rightarrow a + b > c$

Lemma3: We can suppose $a \geq 3$

Proof: (of lemma3)
*Working with integers: $a > 0 \Rightarrow a \geq 1$
*Suppose that: $a = 1$
*We have: $1 = a^n = c^n - b^n = (c-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} \Rightarrow c - b = a = 1$
*This contradicting lemma 2, we have: $a \geq 2$
*Suppose that: $a = 2 \Rightarrow b > 2$ and $c > 2$
*We have:
$a = 2$ and $n \geq 2$ and $c - b > 0$ and $b, c > 2 \Rightarrow 2^n = (c-b)\sum_{k=0}^{n-1} c^k b^{n-1-k} > (c-b)\sum_{k=0}^{n-1} 2^k 2^{n-1-k} = n(c-b)2^{n-1} > 2^n(c-b) \Rightarrow 0 < c - b < 1$
*This being impossible because $c - b$ is an integer, we have well $a \neq 2$
*That is:$a \geq 3$

Lemma4: For $k \geq 2$ we have:
(1)The function $u(t) = t^{-\frac{n\pi}{4}} (\ln(t))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} (t \geq 1)$ is strictly decreasing for $t \geq e^{\frac{1}{n}(\frac{\pi}{4}+\frac{1}{k})}$.
(2) We have: $c > b > a \geq 3 > e^{\frac{1}{n}(\frac{\pi}{4}+\frac{1}{k})} \Rightarrow u(a) > u(b) > u(c)$.
(3)If $\lambda(k) \in]1 - \frac{\pi}{4}, 1[$ is as defined in the lemma 6 below, the function $v(t) = t^{-2\tan(1-\lambda(k))} (\ln(t))^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} (t \geq 1)$ is strictly decreasing for $t \geq e^{\frac{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}{2\tan(1-\lambda(k))}}$.
(4)$\exists p \geq 2$ Such that: $3 \geq e^{\frac{\frac{\pi}{4}(1-\lambda(p)+\frac{1}{p})}{2\tan(1-\lambda(p))}}$.
(5)$p \geq 2$, being the number given by the assertion (4) of lemma4, we have:

$$c > b > a \geq 3 \geq e^{\frac{\frac{\pi}{4}(1-\lambda(p)+\frac{1}{p})}{2\tan(1-\lambda(p))}} \Rightarrow v(a) > v(b) > v(c).$$

Proof: (of lemma4)
(1) *We have: $u'(t) = t^{-n\frac{\pi}{4}-1} (\ln(t))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})-1} \ln\left(t^{-n\frac{\pi}{4}} e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right)$.
*We have:u Strictly decreasing$\Leftrightarrow \ln\left(t^{-n\frac{\pi}{4}} e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right) < 0 \Leftrightarrow t^{-n\frac{\pi}{4}} e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} < 1$.
$\Leftrightarrow t^{n\frac{\pi}{4}} > e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} \Leftrightarrow t > e^{\frac{1}{n}(\frac{\pi}{4}+\frac{1}{k})}$.
(2)*We have: $k, n \geq 2 \Rightarrow \frac{1}{2}(\frac{\pi}{4}+\frac{1}{2}) \geq \frac{1}{n}(\frac{\pi}{4}+\frac{1}{k}) > 0 \Rightarrow 3 > e^{\frac{1}{2}(\frac{\pi}{4}+\frac{1}{2})} = 1.9 \ldots \geq e^{\frac{1}{n}(\frac{\pi}{4}+\frac{1}{k})}$.
*So: $c > b > a \geq 3 > e^{\frac{1}{n}(\frac{\pi}{4}+\frac{1}{k})} \Rightarrow u(a) > u(b) > u(c)$.
(3)*We have:
$v'(t) = t^{-2\tan(1-\lambda(k))-1} (\ln(t))^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})-1} \ln\left(t^{-2\tan(1-\lambda(k))} e^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}\right)$.
*We have:$t \to v(t)$ Strictly decreasing$\Leftrightarrow \ln\left(t^{-2(\tan(1-\lambda(k)))} e^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}\right) < 0$.
$\Leftrightarrow t^{-2(\tan(1-\lambda(k)))} e^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} < 1 \Leftrightarrow t^{2(\tan(1-\lambda(k)))} > e^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} \Leftrightarrow t > e^{\frac{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}{2\tan(1-\lambda(k))}}$.
(4)*Suppose contrarily that: $\forall k \geq 2: e^{\frac{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}{2\tan(1-\lambda(k))}} \geq 3$.
*Tending: $k \to +\infty$, we have, by the Bolzano-Weierstrass theorem:
$\lim_{k\to+\infty} \lambda(k) = \lambda \in [1 - \frac{\pi}{4}, 1]$ and $\frac{1-\lambda}{\tan(1-\lambda)} \leq 1 \Rightarrow e^{\frac{\pi}{8}} = 1.48 \ldots \geq e^{\frac{\frac{\pi}{4}(1-\lambda)}{2\tan(1-\lambda)}} \geq 3$.
*This being impossible, the result follows.
(5)The result follows, from the assertion (4) of lemma4.

Lemma5: We have:

(1) $\lim_{t\to 0}\frac{t}{\tan(t)}=1$.

(2)$\forall t\in\left[0,\frac{\pi}{2}\right]\ \varphi(t)=\tan(t)-t\geq 0$.

(3)$\forall t\in\left[0,\frac{\pi}{4}\right]\ \tau(t)=t-\frac{\pi}{4}\tan(t)\geq 0$. (I.e.:$0\leq t\leq\frac{\pi}{4}\Rightarrow 1\leq\frac{\tan(t)}{t}\leq\frac{4}{\pi}$)

Proof: (of lemma 5)

(1)By the L'Hôpital rule, we have: $\lim_{t\to 0}\frac{t}{\tan(t)}=FI\frac{0}{0}=\lim_{t\to 0}\frac{t'}{(\tan(t))'}=\lim_{t\to 0}\frac{1}{1+(\tan(t))^2}=\frac{1}{1+(\tan(0))^2}=1$.

(2)We have: $\varphi'(t)=1+(\tan(t))^2-1=(\tan(t))^2\geq 0\quad\forall t\in[0,\frac{\pi}{2}]\Rightarrow\varphi$ increasing on $\left[0,\frac{\pi}{2}\right]\Rightarrow\forall t\in[0,\frac{\pi}{2}]\varphi(t)=\tan(t)-t\geq\varphi(0)=0$.

(3)*We have: $\tau'(t)=1-\frac{\pi}{4}(1+(\tan(t))^2)$.

*$\tau'(t)=0, t\in\left[0,\frac{\pi}{4}\right]\Leftrightarrow t=\alpha=\arctan\left(\sqrt{\frac{4}{\pi}-1}\right)\in\left[0,\frac{\pi}{4}\right]$.

* On $[0,\alpha]$: τ is increasing and on $[\alpha,\frac{\pi}{4}]$: τ is decreasing.

*So:

** $\forall t\in[0,\alpha]\ \tau(t)=t-\frac{\pi}{4}\tan(t)\geq\tau(0)=0$.

**$\forall t\in\left[\alpha,\frac{\pi}{4}\right]\ \tau(t)=t-\frac{\pi}{4}\tan(t)\geq\tau\left(\frac{\pi}{4}\right)=0$.

*The result follows because: $\left[0,\frac{\pi}{4}\right]=[0,\alpha]\cup\left[\alpha,\frac{\pi}{4}\right]$.

Lemma 6: We have:

(i) $\forall k\geq 2\exists\lambda(k)\in]1-\frac{\pi}{4},1[$ Such that:

$$\left(\frac{a}{c}\right)^{n\lambda(k)}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}+\left(\frac{b}{c}\right)^{n\lambda(k)}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})}=1$$

(ii) $\lim_{k\to+\infty}\lambda(k)=\lambda\in\left[1-\frac{\pi}{4},1\right]$.

Proof: (of lemma6)

(i)* consider, for $k\geq 2$, the continuous function defined on $[1-\frac{\pi}{4},1]$ by:

$$f(t)=\left(\frac{a}{c}\right)^{nt}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(1-t+\frac{1}{k})}+\left(\frac{b}{c}\right)^{nt}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(1-t+\frac{1}{k})}-1$$

*By the assertion (2) of lemma4, we have:

**$3\leq a<b<c\Rightarrow f\left(1-\frac{\pi}{4}\right)=\left(\frac{a}{c}\right)^{n(1-\frac{\pi}{4})}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}+\left(\frac{b}{c}\right)^{n(1-\frac{\pi}{4})}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}-1$

$=\left(\frac{a}{c}\right)^{n(1-\frac{\pi}{4})}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}+\left(\frac{b}{c}\right)^{n(1-\frac{\pi}{4})}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}-\left(\left(\frac{a}{c}\right)^n+\left(\frac{b}{c}\right)^n\right)$

$=\left(\frac{a}{c}\right)^n\left(\left(\frac{a}{c}\right)^{-n\frac{\pi}{4}}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}-1\right)+\left(\frac{b}{c}\right)^n\left(\left(\frac{b}{c}\right)^{-n\frac{\pi}{4}}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}-1\right)$

$=\left(\frac{a}{c}\right)^n\frac{u(a)-u(c)}{c^{-n\frac{\pi}{4}}(\ln(c))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}}+\left(\frac{b}{c}\right)^n\frac{u(b)-u(c)}{c^{-n\frac{\pi}{4}}(\ln(c))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}}$

>0.

** We have:

$$\begin{cases}1<\ln(3)\leq\ln(a)<\ln(b)<\ln(c)\\ \frac{n}{k}>0\end{cases}\Rightarrow f(1)=\left(\frac{a}{c}\right)^n\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{n\pi}{4k}}+\left(\frac{b}{c}\right)^n\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{n\pi}{4k}}-1$$

$=\left(\frac{a}{c}\right)^n\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{n\pi}{4k}}+\left(\frac{b}{c}\right)^n\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{n\pi}{4k}}-\left(\left(\frac{a}{c}\right)^n+\left(\frac{b}{c}\right)^n\right)$

$=\left(\frac{a}{c}\right)^n\left(\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{n\pi}{4k}}-1\right)+\left(\frac{b}{c}\right)^n\left(\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{n\pi}{4k}}-1\right)<0$.

*So, by the intermediate value theorem, we have:

$$f\left(1-\frac{\pi}{4}\right)f(1)<0 \Rightarrow \forall k\geq 2 \exists \lambda(k)\in]1-\frac{\pi}{4},1[\text{ such that: } f(\lambda(k))=0$$

Lemma7: If $p, \lambda(p)$ are the numbers given respectively by the assertion (4) of lemma 4 and the assertion (i) of lemma6, we have: $\lambda(p)\geq 1-\frac{2}{n}\tan(1-\lambda(p))$, that is: $n\leq \frac{2\tan(1-\lambda(p))}{1-\lambda(p)}$

Proof: (of lemma 7)

Remark: we have:

$$\begin{cases}\lambda(p)\in]1-\frac{\pi}{4},1[\\ p\geq 2\\ n\geq 2\end{cases} \Rightarrow 0<\frac{2}{n}\tan(1-\lambda(p))<\tan\left(\frac{\pi}{4}\right)=1$$

* If not: $\lambda(p)<1-\frac{2}{n}\tan(1-\lambda(p))$

* So, by the assertion (5) of lemma 4, we have:

$$0=\left(\frac{a}{c}\right)^{n\lambda(p)}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}+\left(\frac{b}{c}\right)^{n\lambda(p)}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}-1$$

$$>\left(\frac{a}{c}\right)^{n-2\tan(1-\lambda(p))}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}+\left(\frac{b}{c}\right)^{n-2\tan\left(1-\lambda(p)+\frac{1}{p}\right)}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}-1$$

$$=\left(\frac{a}{c}\right)^{n-2\tan(1-\lambda(p))}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}+\left(\frac{b}{c}\right)^{n-2\tan(1-\lambda(p))}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}-\left(\frac{a^n}{c^n}+\frac{b^n}{c^n}\right)$$

$$=\frac{a^n}{c^n}\left(\left(\frac{a}{c}\right)^{-2\tan(1-\lambda(p))}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}-1\right)+\frac{b^n}{c^n}\left(\left(\frac{a}{c}\right)^{-2\tan(1-\lambda(p))}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}-1\right)$$

$$=\frac{a^n}{c^n}\left(\frac{v(a)-v(c)}{(\ln(c))^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}c^{-2\tan(1-\lambda(p))}}\right)+\frac{b^n}{c^n}\left(\frac{v(b)-v(c)}{(\ln(c))^{\frac{\pi}{4}\left(1-\lambda(p)+\frac{1}{p}\right)}c^{-2\tan(1-\lambda(p))}}\right)$$

>0.

*The obtained relation "0<0" being impossible, we have well: $\lambda(p)\geq 1-\frac{2}{n}\tan(1-\lambda(p))$ (That is: $n\leq \frac{2\tan(1-\lambda(p))}{1-\lambda(p)}$)

RETURN TO THE PROOF OF THEOREM:

*By the assertion (3) of lemma 5 and lemma 7, we have:

$$0<1-\lambda(p)<\frac{\pi}{4}\Rightarrow 2\leq n\leq \frac{2\tan(1-\lambda(p))}{1-\lambda(p)}\leq 2\left(\frac{4}{\pi}\right)=\frac{8}{\pi}=2.546.. \Rightarrow n=2$$

*This ends the proof of the Fermat last theorem.

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