GLOBAL JOURNAL OF ADVANCED ENGINEERING TECHNOLOGIES AND SCIENCES

SOLVABILITY OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH MULTIPLE CHARACTERISTICS

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DOI:10.5281/zenodo.1195107

ABSTRACT

The paper studied the matters of solvability of the second order operator differential equations with multiple characteristics on the semi-axis. In the space of Sobolev type W_2^2 (R_+ ; H) identified conditions that ensured the unique and well-posed solvability are of the considered equation which is cleared by its operator coefficients

KEYWORDS: *solvability condition, boundary-value problem, Hilbert space, self-adjoint operator, Sobolev type* W_2^2 *(R_+ ; H).*

INTRODUCTION

In aseparable Hilbert Space H, we consider a polynomial operator pencil of the second order

$$P(\lambda) = (\lambda E + A)^2 + \lambda A_1, \qquad (1)$$

Where E is the identity operator, A is a self-adjoin positively defined operator, and $A_1, A_2$are linear operator unbounded. This equation can associate a polynomial pencil

$$Q\left(\frac{d}{dt}\right)u\ (t) = f(t) \qquad (2)$$

In space L_2 (R;H), we denote by L_2 (R;H)(see. [1]) Hilbert space of all vector valued functions defined in R_+ = [0,+∞) with values ion H, which have the norm

$$\|f\|_{L_2(R;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2\, dt\right)^{1/2} \prec \infty.$$

Now we introduce the following set:

$$W_2^2(R;H) = \left\{u(t): \frac{d^2u(t)}{dt^2} \in L_2(R;H), A^2u(t) \in L_2(R;H)\right\}.$$

With the norm

$$\|u\|_{W_2^2([a,b];H)} = \left(\left\|\frac{d^2u}{dt^2}\right\|_{L_2([a,b]:h)}^2 + \|A^2u\|_{L_2([a,b]:h)}^2\right)^{1/2},$$

MAIN RESULTS

Definition 1. If $f(t) \in L_2(R;H)$ there exists vector valued function $u(t) \in w_2^2(R;H)$, satisfying equation (2) almost everywhere, then it is called a regular solution of equation (2).

Definition 2. If for any $f(t) \in L_2(R;H)$ there exists a regular solution $u(t)$ of equation (2), and holds the inequality.

$$\|u\|_{W_2^2(R;H)} \leq const\|f\|_{L_2(R;H)}$$

Then we will call the equation (2) regularly solvable.

In the paper we will get the conditions, expressed in the operator coefficients of equation (1), we proved the regular solvability of equation (2).

Let's introduce the following notations. Calling $Q_0$and $Q_1$the operators acting from the space $w_2^2(R;H)$as follows:

$$Q_0u(t) \equiv (\frac{d}{dt} + A)^2\, u(t), u(t) \in w_2^2(R_+;H)$$

$$Q_1u(t) \equiv B\frac{du(t)}{dt}, u(t) \in w_2^2(R_+;H)$$

Theorem 1. The operator $Q_0$is an isomorphism from the space $w_2^2(R;H)$ to the space $L_2(R;H)$

Proof. Taking into account the theorem of intermediate derivatives [2], it is easy to prove that the operator Q_0 acts from $w_2^2(R;H)$to $L_2(R;H)$ be bounded. Using Fourier transforms for the equation $Q_0u(t)=f(t), f(t)\in L_2(R;H), u(t)\in w_2^2(R_+;H)$, we obtain

$$(i\xi E+A)^2\check{u}(\xi)=\check{f}(\xi),$$

where $\tilde{u}(\xi)$, $\tilde{f}(\xi)$ Fourier transform for the functions u(t), f(t), respectively the operator pencil (i ξE +A)^2 is invertible and moreover

$$\tilde{u}(\xi)=(i\,\xi E+A)^{-2}\tilde{f}(\xi). \quad (3)$$

Hence,

$$u(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(i\xi E+A)^{-2}\left(\int_0^{+\infty}f(s)e^{-i\zeta t}ds\right)e^{i\zeta t}d\zeta \text{ , } t\in R.$$

we show that u(t) ∈ w_2^2(R$_+$; H) . by using the parseval equality and (3), we obtain

$$\|u\|^2_{W_2^2(R_+;H)}=\left\|\frac{d^2u}{dt^2}\right\|^2_{L_2R_+;H}+\|A^2u\|^2_{L_2R_+;H}=$$

$$=\left\|-\zeta^2\hat{u}(\xi)\right\|^2_{L_2(R_+;H)}+\left\|A^2\hat{u}(\xi)\right\|^2_{L_2(R_+;H)}=$$

$$=\left\|-\zeta^2(i\zeta E+A)^{-2}f(\zeta)\right\|_{L_2(R_+;H)}+\left\|A^2(i\zeta E+A)^{-2}f(\zeta)\right\|L_2(R_+;H)\le$$

$$\le\sup_{\zeta\varepsilon R}\left\|-\zeta^2(i\zeta E+A)^{-2}\right\|^2_{H\to H}\|f(\zeta)\|^2_{L_2(R_+;H)}+$$

$$+\sup_{\zeta\varepsilon R}\left\|A^2(i\zeta E+A)^{-2}\right\|^2_{H\to H}\|f(\zeta)\|^2_{L_2(R_+;H)} \quad (4)$$

From the spectral decomposition of the operator A (denoting by σ(A) the spectrum of operator A) for $\zeta\in R$ we have

$$\left\|-\zeta^2(i\zeta E+A)^{-2}\right\|=\sup_{\mu\in\sigma(A)}\left|-\zeta^2(i\zeta+\mu)^{-2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\frac{\zeta^2}{\zeta^2+\mu^2}\le 1, \quad (5)$$

$$\left\|A^2(i\zeta E+A)^{-2}\right\|=\sup_{\mu\in\sigma(A)}\left|\mu^2(i\zeta+\mu)^{-2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\frac{\mu^2}{\zeta^2+\mu^2}\le 1. \quad (6)$$

Taking into account (5) and (6) into (4) we obtain :

$$\|u_1\|^2_{W_2^2} \leq \|f(\zeta)\|^2_{L_2(R;H)} = 2\|f(t)\|^2_{L_2(R_+;H)}$$

. Consequently, $u(t) W_2^2(R_+;H)$. . We finish the proof of the theorem by taking into account the Banach theorem of theinverseoperator.

Theorem 2. Let the operators $A_j A^{-j}$, $j = 1,2$ are bounded on H .Then The operator Q_1 acting from the space $W_2^2(R_+;H)$ to $L_2(R_+;H)$ is bounded.

Proof. Since $u(t) \in W_2^2(R_+;H)$ then bu applying the theorem on intermediate derives $[2]$, we have

$$\|Q_1 u\|_{L_2(R_+;H)} \leq \|A_1 A^{-1}\|_{H\to H} \left\|A \frac{du}{dt}\right\|_{L_2(R_+;H)} \leq$$
$$\leq const\|u\|_{W_2^2(R_+;H)}$$

The theorem is proved

Theorem (1) show that the norm $\|Q_1 u\|_{L_2(R_+;H)}$equivalent in $W_2^2(R_+;H)$to the norm $\|u\|_{W_2^2(R_+;H)}$ therefore, by theorem of intermediate derivatives [2] the number $N_1 = sup_{0\neq W_2^2(R_+;H)} \left\|A^j \frac{d^{3-j}u}{dt^{3-j}}\right\|_{L_2(R_+;H)} \|Q_0 u\|^{-1}{}_{L_2(R_+;H)}$,is finite.

Let us estimate these values

Theorem 3.

$$N_1 \leq c_1 \text{where} c_1 = \frac{1}{2}$$

Proof

Let $Q_0 u(t) = f(t)$and using the Fourier transform we obtain

$$\left\|A\,(i\xi)(i\xi E + A)^{-2}\tilde{f}(\xi)\right\|_{L_2(R_+;H)} \leq sup_{\xi\in R}\|A\,(i\xi)(i\xi E + A)^{-2}\|_{H\to H}\left\|\tilde{f}(\xi)\right\|_{L_2(R_+;H)} \quad (7)$$

Taking into account the spectral decomposition of operator A, we estimate for $\xi \in R$ the following norms:

$$\left\|A\,(i\zeta)(i\zeta E + A)^{-2}\right\| = \sup_{\mu\varepsilon\sigma(A)} \left|\mu\,(i\zeta)(i\zeta+\mu)^{-2}\right| \leq$$

$$\leq \sup_{t=\frac{\zeta^2}{\mu^2}\geq 0} \frac{t^{\frac{1}{2}}}{(t+1)} = c_1 \quad (8)$$

Substituting from (8) in (7), we have

$$\left\|A\,(i\zeta)\,(i\zeta E + A)^{-2} f(\zeta)\right\|_{L_2(R_+;H)} \leq c_1 \left\|\hat{f}(\zeta)\right\|_{L_2(R_+;H},$$

Are equivalent in its turn to the inequalities

$$\left\|A \frac{d\,u}{dt}\right\|_{L_2(R_+;H)} \leq c_1 \|Q_0 u\|_{L_2(R_+;H)},$$

The theorem is proved.

Theorem 4. Let the operators $A_1 A^{-1}$ defined in Theorem3. Then equation inequality $c_1 \left\| A_1 A^{-1} \right\|_{H \to H} < 1$,

Where c_1 is defined in Theorem 3. Then the equation (2) regularly solvable.

Poof . Cleary that equation (2) can be Written as

$$Q_0 u(t) + Q_1 u(t) = f(t), \qquad (9)$$

Where $f(t) \in L_2(R_+; H), u(t) \in W_2^3 R_+; H$. The operator Q_0 by Theorem (1) there exists a bounded inverse, which acts from $L_2(R_+; H)$within$W_2^2(R_+; H)$. Then after replacing $Q_0 u(t) = v(t)$ equation (9) can be written as

$$(E + Q_0 Q_0^{-1}) v(t) = f(t).$$

Now we prove under the theorem conditions that the norm $\left\| Q_1 Q_0^{-1} \right\|_{L_2(R_+;H) \to L_2(R_+;H)} < 1.$

By theorem (3), we have

$$\left\| Q_1 Q_0^{-1} v \right\|_{L_2(R_+;H)} = \left\| Q_1 u \right\|_{L_2(R_+;H)} \le \left\| A_1 A^{-1} \right\|_{H \to H} \left\| A \frac{d\ u}{dt} \right\|_{L_2(R_+;H)} \le$$

$$\le \sum_{j=1}^{2} c_1 \left\| A_1 A^{-1} \right\|_{H \to H} \left\| Q_0 u \right\|_{L_2(R_+ H)} = c_1 \left\| A_1 A^{-1} \right\|_{H \to H} \|v\|_{L_2(R_+;H)}$$

Consequently,

$$\left\| Q_1 Q_0^{-1} \right\|_{L_2(R_+;H) \to L_2(R_+;H)} \le c_1 \left\| A_1 A^{-1} \right\|_{H \to H} < 1.$$

Thus, the operator $E + Q_1 Q_0^{-1}$ is invertible in $L_2(R_+; H)$ and hence u(t) can be determined by u(t)= $Q_0^{-1} \left(E + Q_1 Q_0^{-1} \right)^{-1} f(t)$, moreover

$$\|u\|_{W_2^2(R_+;H)} \le \left\| P_0^{-1} \right\|_{L_2(R_+;H) \to W_2^2(R_+;H)} \left\| \left(E + Q_1 Q_0^{-1} \right) \right\|_{L_2(R_+;H) \to L_2(R_+;H)} \|f\|_{L_2(R_+;H)} \le const \|f\|_{L_2(R_+;H)}..$$

The theorem is proved.

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