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CONFIRMATION OFTHE BEAL-BRUN-TIJDEMAN-ZAGIER CONJECTURE

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ABSTRACT

The present work confirms the Beal's conjecture, remained open since 1914 and saying that: "the Diophantine equation $a^x + b^y = c^z$ (called « the Fermat generalized equation » or « the Fermat-Catalan equation ») has no solution in $\mathbb{N}^*$ for: $x > 2, y > 2, z > 2$ with primitive integers a, b, c". I say that the present work confirms the Beal conjecture by using elementary tools of mathematics, such as the L'Hôpital rule and the intermediate value theorem. The proof uses also the Catalan-Mihailescu theorem, the growth properties of some elementary functions and some methods developed in my paper on the Fermat last theorem [13] published by the GJAETS in 10/12/2018.

KEYWORDS: A Diophantine equation-The Beal conjecture-The Fermat little theorem-The Fermat last theorem -Primitive integers-intermediate value theorem- Catalan/Mihailescu theorem-L'Hôpital rule
2010 Mathematics Subject Classification: 11 A xx (Elementary Number theory).

INTRODUCTION

DEFINITION: We call « the Tijdeman and Zagier conjecture » or « Beal conjecture » or what I call « the Beal-Brun-Tijdeman-Zagier conjecture» the following assertion : « the Diophantine equation $a^x + b^y = c^z$ (called « the Fermat generalized equation » or « the Fermat-Catalan equation ») has no solution in $\mathbb{N}^*$ for : $x > 2, y > 2, z > 2$ and $gdc(a, b, c) = 1$(we say that : a, b, c are primitive)», $gcd(a, b, c)$ denoting the greatest common divisor of the natural integers a, b and c.

Remark: 1) The case $x = y = z = n$ is the Fermat last theorem which I have showed in 10/12/2018 (see [13]) by an elementary short proof and also proved in a hard, relatively long, proof in 1994 (see [28]) by A .Wiles.
2) The condition $gcd(a, b, c) = 1$ is done there to avoid trivialities. Indeed, going from the Diophantine equation: $a^x + b^y = c^z$ multiplied by $a^{21x}b^{14y}c^{6z}$, we obtain an infinite number of solutions of the Diophantine equation: $A^2 + B^3 = C^7$ as: $(a^{11x}b^{7y}c^{3z})^2 + (a^{7x}b^{5y}c^{2z})^3 = (a^{3x}b^{2y}c^z)^7$
Recall that the Diophantine equation $A^2 + B^3 = C^7$ was completely resolved, in [16], where Poonen-Schaefer-Stoll showed that the strictly positive entire primitive solutions are $(A, B, C) = (2213459\,, 1414\,, 65)$ And $(A, B, C) = (15312283\,, 9262\,, 113)$
3) Recall that, if $\zeta(3) = \sum_{k=1}^{+\infty} \frac{1}{k^3}$ denotes the Apéry's constant, the probability for three integers to be primitive is $\frac{1}{\zeta(3)}$.
4) Poonen-Schaefer-Stoll remarked in [16] that: for any integers (a, b, c) with $|a| \le K^{\frac{1}{x}}, |b| \le K^{\frac{1}{y}}, |c| \le K^{\frac{1}{z}}$ the probability to have: $a^x + b^y = c^z$ is$1/K$.

Some History: *the conjecture was formulated independently by the banker and amateur mathematician Andrew Beal [18] in 1993 and the mathematicians Robert Tijdeman and Don Zagier in 1994. But it seems that it has appeared in the Brun works since 1914(see [8]). Andrew Beal [18] devoted, since 1997, an increasing price for any one can prove or disapprove his conjecture.
*In 300 before Jesus Christ, Euclid resolved completely, in ([11], book x), the Diophantine equation $a^2 + b^2 = c^2$ by giving its general solutions (See proposition4 below).
*In the years 1600 Fermat showed that the Diophantine equation: $a^2 + b^4 = c^4$ has no solution.
*Then it was showed that the Diophantine equation: $a^x + b^4 = c^4$ has no solution for any integer x.
*In 1994, Wiles [28] showed by a relatively long proof of about 100 pages that the Diophantine equation:
$a^n + b^n = c^n$ has no solution with not null integers a, b, c for $n > 2$ by using powerful tools of number theory. This is the Fermat last theorem.

*In 2002, Preda Mihailescu [14], [15] showed that Diophantine equation: $1 + b^y = c^z$ has the sole solution $(b, c, y, z) = (2,3,3,2)$ (Resolving, so, what is known as the « Catalan conjecture»). The proof uses cyclotomic Fields and Galois modules.
*In 2005 Bjorn Poonen, Edward F. Schaefer and Michael Stoll [16] showed that the Diophantine equations: $a^x + b^y = c^z$ with $\{x, y, z\}$ = all permutations of $(2,3,7)$ have only 4 solutions with no power>2.
*In 2009, David Brown [4] studied the case: $(x, y, z) = (2,3,10)$
*In 2009, Michael Bennet, Jordan Ellenberg and Nathan Ng studied [1] the case: $(x, y, z) = (2,4, n)$ pour $n \geq 4$.
*In 2014, Samir Siksek and Michael Stoll studied [17] the case $(x, y, z) = (2,3,15)$
*In 2018, M.Ghanim showed in [13] the Fermat last theorem, which is a special case of the Beal conjecture, by an elementary short proof.
For more History see[2], [3] and [19].

The note: The purpose of the present short note is to give a relatively elementary proof of the Beal conjecture based essentially, on the intermediate value theorem, the L'Hôpital rule, the Catalan-Mihailescu theorem, the growth properties of some elementary functions and some methods of [13].
The note is organized as follows. The §1 is an introduction giving the necessary definitions and some History. The §2 is devoted to the preliminaries where we give some remarks and the results needed in the proofs of our main results. The §3 gives the proof of the Beal conjecture. The §4 gives some references for further reading.

Results: Our main result is:

Theorem: (proving the Beal's conjecture) let x, y *and* z three integers≥ 2 . If: $n = \min(x, y, z)$, then:

$$\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^x + b^y = c^z \text{ and } gcd(a, b, c) = 1 \Rightarrow n = 2$$

Methods: The methods used in the paper are as follows.
* Going from the integers $x, y, z \geq 2$ and $a, b, c \in \mathbb{N}^*$ such that $gcd\ (a, b, c) = 1$, and $a^x + b^y = c^z$, I show that $n = \min(x, y, z) = 2$. For doing this I distinguish the following 2 cases:

First case: $a = 1$.
In this case $n = \min(x, y, z) = 2$ follows from the Catalan-Mihailescu theorem (See Proposition7 below).

Second case: $a \geq 2$
Supposing contrarily that $n = \min(x, y, z) \geq 3$, I show, by the intermediate value theorem, that:

$$\exists \theta \in]0, \frac{1}{2}[\text{ such that: } \frac{1}{2}(\frac{a^x}{c^z})^\theta \frac{\ln(a^x)}{\ln(c^z)} + \theta(\frac{b^y}{c^z})^\theta \frac{\ln(b^y)}{\ln(c^z)} = \frac{1}{2}$$

Then I prove, for this θ, that the integer $n = \min(x, y, z)$ satisfies: $n < \frac{3}{2}(\frac{1}{1-\theta})$

So : $3 \leq n < \frac{3}{2}\left(\frac{1}{1-\theta}\right) < \frac{3}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 3$ is a contradiction showing that : $n = 2$.

PRELIMINARIES

Below are given some remarks and the results needed for showing our main theorem.
Proposition1: (properties of the greatest common divisor) [26] $\gcd(a, b, c)$ is the strictly positive greatest common divisor of the integers a, b, c. If $\gcd(a, b, c) = 1$ we say that "a, b, c are primitive or coprime". The gcd has the below properties:
(i)$d = \gcd(a, b, c) \Rightarrow d > 0$ and d divides the three integers a, b, c
(ii)d divides the three integers a, b, c and $d>0 \Rightarrow d \leq \gcd(a, b, c)$
(iii)d divides the three integers $a, b, c \Rightarrow d$ divides $\gcd\ (a, b, c)$
(iv) (Bezout theorem [25]) $\gcd(a, b, c) = 1 \Leftrightarrow \exists u; v, w \in \mathbb{Z}\ ua + vb + wc = 1$
(v)$\gcd(a, b, c) = d \Leftrightarrow \exists u, v, w \in \mathbb{N}$ Such that $\begin{cases} a = ud, b = vd, c = wd \\ \gcd(u, v, w) = 1 \end{cases}$
(vi)* $\gcd(a, b, c) = d \Rightarrow \exists u, v, w \in \mathbb{Z}\ ua + vb + wc = d$
*The reciprocal implication is not always true
(vii)$\gcd(a, b, c) = 1 \Leftrightarrow \forall n, m, p \in \mathbb{N}^*\ \gcd(a^n, b^m, c^p) = 1$

Proposition2: (The Gauss theorem) [20] if p is a prime integer then p is a divisor of the integer $c^z \Rightarrow p$ is a divisor of the integer c. Recall that p is a prime integer if its set of divisors is $\{1,p\}$.

Proposition3: (The fundamental theorem of arithmetic) [22] $\forall n \in \mathbb{N}^* - \{1\} \exists p_1, p_2, \dots, p_m$ prime integers $\exists \alpha_1, \alpha_2 \dots, \alpha_m \in \mathbb{N}^*$ such that: $n = \prod_{i=1}^{m} p_i^{\alpha_i}$
In Particular: $\forall n \in \mathbb{N}^* - \{1\} \exists p$ a prime number such that p divides n i.e. $\exists N \in \mathbb{N}^*$ such that: $n = pN$

Proposition4: (Euclid (was in life about 300 before J.C)) (See [11], book X) the Diophantine equation: $a^2 + b^2 = c^2$, has the particular solution: $(a, b, c) = (3,4,5)$ and has for general solutions: $\begin{cases} a = 2xyz \\ b = x(z^2 - y^2) \\ c = x(z^2 + y^2) \end{cases}$
With: $(x, y, z) \in \{(p, q, r) \in \mathbb{N}^3$ such that $r > q$ and p, q, r are of different parity$\}$

Proposition5: For: (x, y, z) =all the permutations of {2, 4, 4}, the Diophantine equation: $a^x + b^y = c^z$ has no solution in $\mathbb{N}^*$.

Proof: (of proposition5)
* For the Diophantine equation $a^2 + b^4 = c^4$ we have: $a^2 + (b^2)^2 = (c^2)^2$
*So, by proposition4: $a = 3, b^2 = 4$ and $c^2 = 5$ is a solution i.e. $a = 3, b = 2$ and $c = \sqrt{5}$
*But $\sqrt{5} \notin \mathbb{Q} \Rightarrow c \notin \mathbb{N}$
*This being impossible the result is showed.

Proposition6: (Euler theorem) [12] The Diophantine $a^3 + b^3 = c^3$ has no solutions in $\mathbb{N}^*$.

Proposition7 :(Eugène Charles Catalan-Preda Mihailescu theorem) [14], [15] The Diophantine equation: $1 + b^y = c^z$ (with: b, c, z, y integers > 1) has the sole solution: $b = 2, c = 3, y = 3$ and $z = 2$.

Proposition8 : (The Fermat last theorem) [13], [29] We have :

$$\exists a, b, c \in \mathbb{N}^* \text{ such that } 0 < a < b < c \text{ and } a^n + b^n = c^n \Rightarrow n = 2$$

Proposition9 :(Poonen-Schaefer-Stoll [16]) for (x, y, z) = all the permutations of {2, 3, 6} the sole case for which the Diophantine equation $a^x + b^y = c^z$ has non trivial solutions is $x = 6, a = 1, y = 3, b = 2, z = 2, c = 3$

Proposition10: (Beukers theorem [3]) in the case $(x, y, z) = (2,2, z)$ with:$z \geq 2$ or $(x, y, z) = (2,3,3), (2,3,4), (2,3,5)$, the set of solutions of the Diophantine equation $a^x + b^y = c^z$ is empty or infinite.

Proposition11: (Darmon-Granville theorem [10]) for any fixed choice of positive integers x, y, z satisfying the hyperbolic case:$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, only finitely many primitive triples (a, b, c) solving the Diophantine equation $a^x + b^y = c^z$ exist.

Note that this result resolves partially the Fermat-Catalan conjecture which is stronger because allows the exponents x, y, z to vary.

Proposition12: (See [18], p 9) we have:$\forall x, y, z \in \mathbb{N}^*$ $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{41}{42}$

Proposition 13: For any integers $x, y, z \geq 2$, we have:
(1)$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 1 \Rightarrow n = \min(x, y, z) \leq 3$
(2)So: $(x, y, z) =$ $(3,3,3)$ *or* all the permutations of $\{2,4,4\}$ *or* all the permutations of $\{2,3,6\}$or all the permutations of $\{2,2,\mathrm{k}\}(\mathrm{k} \geq 2)$or all the permuations of $\{2,3,3\}$or all the permutations of $\{2,3,4\}$or all the permutations of $\{2,3,5\}$
(3)$n = \min(x, y, z) > 3 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$

Proof: (of proposition 13)
(1)We have: $x \geq n, y \geq n, z \geq n \Rightarrow \frac{1}{n} \geq \frac{1}{x}, \frac{1}{n} \geq \frac{1}{y}, \frac{1}{n} \geq \frac{1}{z} \Rightarrow \frac{3}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 1$
$\Rightarrow 3 \geq n = \min(x, y, z) \geq 2$
(2) The result follows from by a simple calculation, because the assertion (1) of proposition 13 $\Rightarrow n = 2$ *or* $n = 3$.
(3) The result is obtained by considering the contrapositive proposition of the assertion (1) in proposition13 (see proposition23 below).

Proposition14: For the Diophantine equation: $a^x + b^y = c^z$, we can suppose $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$ called the hyperbolic case.

Proof : (of proposition14)
Indeed for the two other cases, we have :
***Second case called the Euclidean case :** $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, a simple analysis shows that :

$(x, y, z) = (3,3,3)$ *or* all the permutations of $\{2,4,4\}$ *or* all the permutations of $\{2,3,6\}$

These cases are completely resolved respectively by proposition5, proposition6 and proposition9 above.
***Third case called the spherical case :** $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1$, a simple analysis shows that :
$(x, y, z) =$ all the permutations of $\{2,2,\mathrm{k}\}(\mathrm{k} \geq 2)$ or all the permuations of $\{2,3,3\}$ or all the permutations of $\{2,3,4\}$ or all the permutations of $\{2,3,5\}$
These cases are completely resolved by the Beukers theorem (proposition10).

Proposition15: From the solutions, (a, b, c, x, y, z), of the Diophantine equation $a^x + b^y = c^z$ with $gcd(a, b, c) = 1$; $x, y, z \geq 2$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, we know the following ten ones :
*$1^6 + 2^3 = 3^2, 2^5 + 7^2 = 3^4$(N. Bruin, 2003[6]), $13^2 + 7^3 = 2^9$(N. Bruin, 2004[7]), $2^7 + 17^3 = 71^2$(Poonen − Schaefer − Stoll, 2005[16]), $3^5 + 11^4 = 122^2$(Bruin, 2003[6])
*$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 + 15312283^2 = 113^7$
(The three discovered by Poonen-Schaefer-Stoll, 2005[16])
$43^8 + 96222^3 = 30042907^2$(Bruin, 2003[6]), $33^8 + 1549034^2 = 15613^3$(Bruin, 1999[5])

Remark: 1) for all the examples of the precedent proposition15, we have: $\min(x, y, z) = 2$.
2) S.Siksek and M.Stoll [17] talk, following H.Darmon [9] and H .Darmon-A. Granville [10], about the generalized Fermat conjecture (concerning the hyperbolic case: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$) which says that the sole non trivial primitive solutions are those cited in the above proposition15.

Proposition16: the hypothesis « $gcd(a, b, c) = 1$ » is a necessary condition in the Beal-Brun-Tijdeman-Zagier conjecture.
Proof: (of proposition16)
(i)For example:
*$2^n + 2^n = 2^{n+1}$, for:$n \geq 0$. Note that: 2 divides $gcd(2^n, 2^n, 2^{n+1}) \neq 1$.
*$3^{3n} + (2.3^n)^3 = 3^{3n+2}$ for $n \geq 1$.
Note that: 3 is a common factor to $a = 3, b = 2.3^n$ and $c = 3$ so $gcd(a, b, c) \neq 1$.
*$(p^n - 1)^{2n} + (p^n - 1)^{2n+1} = (p.(p^n - 1)^2)^n$ for $n \geq 3$ and $p \geq 2$. Note that $a^n - 1$ is a common factor to $a = p^n - 1, b = p^n - 1$ and $c = p(p^n - 1)^2$ so $gcd(a, b, c) \neq 1$
*$(p(p^n + q^n))^n + (q(p^n + q^n))^n = (p^n + q^n)^{n+1}$ for $n \geq 3$ and $p, q \geq 1$.
Note that: $p^n + q^n$ is a common factor to $a = p(p^n + q^n), b = q(p^n + q^n)$ and $c = p^n + q^n$.
(ii) We can, in fact, construct from any solution (a_1, b_1, c_1) such that $a_1^x + b_1^y = c_1^z$ an infinite number of solutions (a_n, b_n, c_n) such that:
1) $a_n^x + b_n^y = c_n^z$.
2) $a_n = a_{n-1}^{yz+1}.b_{n-1}^{yz}.c_{n-1}^{yz}, b_n = a_{n-1}^{xz}.b_{n-1}^{xz+1}.c_{n-1}^{xz}$ And $c_n = a_{n-1}^{xy}.b_{n-1}^{xy}.c_{n-1}^{xy+1}$
3) $gcd(a_n, b_n, c_n) \neq 1$ because $a_{n-1}.b_{n-1}.c_{n-1}$ divides $gcd(a_n, b_n, c_n)$

Proposition17: (The Fermat's little theorem) [21] for any prime integer p and for any positive integer a: $\exists k \in \mathbb{N}\ a^p = a + kp$

Proposition18: For the Diophantine equation: $a^x + b^y = c^z$, we can suppose $x, y, z \geq 2$ to be prime integers.
Proof: (of proposition18)
*By proposition3: $\exists p, q, r$ prime integers $\exists X, Y, Z \in \mathbb{N}^*$ such that: $x = pX, y = qY, z = rZ$
*So: $a^x + b^y = c^z \Leftrightarrow (a^X)^p + (b^Y)^q = (c^Z)^r \Leftrightarrow A^p + B^q = C^r, A = a^X, B = b^Y, C = c^Z$
Proposition19: We have:
$a^x + b^y = c^z$ And $\gcd(x, y, z) \neq 1 \Rightarrow x = 2, y = 2, z = 2, a = 3, b = 4$ *and* $c = 5$
Proof: (of proposition 19)
*$\gcd(x, y, z) \neq 1 \Rightarrow \exists p$ a prime number $\exists X, Y, Z \in \mathbb{N}^*$such that:$x = pX, y = pY, z = pZ$
*So: $a^x + b^y = c^z \Rightarrow (a^X)^p + (b^Y)^p = (c^Z)^p$
*So, the Fermat last theorem (see [13]) and proposition 4$\Rightarrow p = 2, a^X = 3, b^Y = 4, c^Z = 5$
*Finally: Gauss theorem $\Rightarrow X = 1, a = 3, Y = 1, b = 4, Z = 1, c = 5$
Proposition20: if $\gcd(a, b, c) = p \neq 1$, we have:
(1)$a^x + b^y = c^z \Rightarrow \exists \alpha, \beta, \gamma \in \mathbb{N}^*$:$(\beta y - \alpha x)(\gamma z - \alpha x)(\gamma z - \beta y) = 0$
(2) $\exists \delta, \epsilon, \theta \in \mathbb{N}^*$ Such that: p divides $\theta z - \epsilon y - \delta x$
Proof: (of proposition20)
(1)*If $\gcd(a, b, c) \neq 1$, then: $\exists p$ a prime number$\exists \alpha, \beta, \gamma \in \mathbb{N}^* \exists A, B, C \in \mathbb{N}^*$ such that: $a = Ap^\alpha$, $b = Bp^\beta$, $c = Cp^\gamma$ and p is not a divisor of A, B and C.
*We have: $a^x + b^y = c^z \Leftrightarrow p^{\alpha x}A^x + p^{\beta y}B^y = p^{\gamma z}C^z$
First case: $\alpha x < \beta y$
*We have: $A^x + p^{\beta y - \alpha x}B^y = p^{\gamma z - \alpha x}C^z$
*So, we have necessarily: $\gamma z - \alpha x = 0$
Indeed, by the Gauss theorem: $\gamma z - \alpha x < 0 \Rightarrow p$ divides C and $\gamma z - \alpha x > 0 \Rightarrow p$ divides A
Second case: $\alpha x > \beta y$
*We have:$p^{\alpha x - \beta y}A^x + B^y = p^{\gamma z - \beta y}C^z$
* As in the first case, we have necessarily: $\gamma z - \beta y = 0$
Third case: $\alpha x = \beta y$
*So, in any case, we have:$\exists \alpha, \beta, \gamma \in \mathbb{N}^*$ such that:
$\alpha x = \beta y$ Or $\gamma z = \alpha x$ Or $\gamma z = \beta y \Leftrightarrow (\beta y - \alpha x)(\gamma z - \alpha x)(\gamma z - \beta y) = 0$
(2)*By the Fermat's little theorem (see proposition 17): x, y, z being prime integers $\Rightarrow \exists \delta, \epsilon, \theta \in \mathbb{N}^*$ such that: $a^x = a + \delta x, b^y = b + \epsilon y, c^z = c + \theta z$
*So: $a^x + b^y = c^z \Rightarrow a + b - c = p(Ap^{\alpha-1} + Bp^{\beta-1} - Cp^{\gamma-1}) = \theta z - \epsilon y - \delta x$
Proposition 21: For the Diophantine equation: $a^x + b^y = c^z$, we have:
$a = 0$ And $\gcd(a, b, c) = 1 \Rightarrow b = c = 1$
Proof: (of proposition 21)
The case:$a = 0 \Rightarrow b^y = c^z$, so: $\gcd(a, b, c) = 1 \Rightarrow c = b = 1$
Indeed: if $b \neq 1$, by proposition3:$\exists B \in \mathbb{N}^*\ \exists p \geq 2$ a prime integer such that $b = Bp$
So: $A^y p^y = c^z$ and Gauss theorem$\Rightarrow p$ devides $c \Rightarrow \exists C \in \mathbb{N}^*$ such that:$c = Cp$
Then p, being a common devisor for $a = 0, b = Bp, c = Cp$, and $\gcd(a, b, c) = 1$ being the greatest common devisor, we have: p devides 1 i.e. $p = 1$. This contradicting "$p \geq 2$" the assertion is proved
*The case, being completely determined, we can suppose: $a > 0$
Proposition22:(The contrapositive proposition and some logic)) [27] if P, Q are two propositions, and if we note by $non(P)$ the negation of P (For example: $\text{non}(\exists) = \forall, non(=) = \neq, non(P\ and\ Q) = non(P) or\ non(Q)$), we call the contrapositive proposition of the proposition $(P \Rightarrow Q)$ the proposition: $(non(Q) \Rightarrow non(P))$. We have: $(P \Rightarrow Q) \Leftrightarrow (non(Q) \Rightarrow non(P))$. Recall that:

(1) $(P \Rightarrow Q) \Leftrightarrow (non(P) or Q)$
(2) $\left((\forall x \in E)\ \left(P(x) or\ Q(x)\right)\right) \Rightarrow ((\forall x\ P(x)) or\ (\exists x\ Q(x)))$
(3) $((\forall x \in E)\ (\ P(x) or\ Q(x)))$ das not imply $((\forall x \in E\ \ P(x)) or\ (\forall x \in E\ Q(x)))$
(4) $((\forall x \in E)(\ P(x) and\ Q(x))) \Leftrightarrow ((\forall x \in E\ P(x))$ and $(\forall x \in E\ Q(x)))$
(5) $\left((\exists x \in E)(P(x) and\ Q(x))\right) \Rightarrow ((\exists x \in E\ P(x)) and\ (\exists x \in E\ Q(x)))$
(6) $((\exists x \in E\ P(x)) and\ (\exists x \in E\ Q(x)))$ das not imply $\left((\exists x \in E)(P(x) and\ Q(x))\right)$

(7) $((\exists x \in E)(P(x) or\ Q(x))) \Leftrightarrow ((\exists x \in E\ P(x)) or\ (\exists x \in E\ Q(x)))$

Proposition23: (The intermediate value theorem) [23] Let $\varphi: [a,b] \to \mathbb{R}$ (with: $a < b$) a continuous function, then :$\varphi(a)\varphi(b) < 0 \Rightarrow \exists c \in]a,b[$ such that $\varphi(c) = 0$

Recall that:

(i)$\lim_{x \to a} f(x) = l \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0$ such that: $|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$

(ii) The function f is continuous in the point $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

(iii) The function f is continuous on an interval $I \Leftrightarrow f$ is continuous in any point $a \in I$

Proposition24: (L'Hôpital rule) [24]Let f and g two continuous functions on$]a,b]$ $(a < b)$, having a derivative on $]a,b[$ and such that: $\lim_{t \to b^-} f(t) = \lim_{t \to b^-} g(t) = 0$ (we call that we are in an indeterminate form (IF)$\frac{0}{0}$) or $\lim_{t \to b^-} f(t) = \pm\infty$, $\lim_{t \to b^-} g(t) = \pm\infty$ (we call that we are in an indeterminate form (IF)$\frac{\infty}{\infty}$) then : $\lim_{t \to b^-} \frac{f(t)}{g(t)} = \lim_{t \to b^-} \frac{f'(t)}{g'(t)}$. The process is repeated if $\lim_{t \to b^-} \frac{f'(t)}{g'(t)}$ is, also, an IF $\frac{0}{0}$ or an IF $\frac{\infty}{\infty}$ … and so on until the determination.

Recall that :

(i)The function f is derivable in the point $a \Leftrightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \in \mathbb{R}$

(ii) $(f^n)' = nf'f^{n-1}$ (iii)$(\ln(|f|))' = \frac{f'}{f}$

(iii)If we mean by an increasing function on the interval I a function f such that : $\forall x, y \in I: x < y \Rightarrow f(x) < f(y)$ and by a decreasing one a function g such that $\forall x, y \in I: x < y \Rightarrow g(y) < g(x)$ we have for derivable functions f and g :

(a)f strictly increasing on $I \Leftrightarrow \forall x \in I\ \ f'(x) > 0$

(b) g strictly decreasing on $I \Leftrightarrow \forall x \in I\ \ g'(x) < 0$

THE PROOF OF BEAL CONJECTURE

Theorem: (Proving the Beal's conjecture) let x, y and z three integers≥ 2 .

$$\exists a,b,c \in \mathbb{N}^* \text{ such that : } a^x + b^y = c^z \text{ and } gcd(a,b,c) = 1 \Rightarrow \min(x,y,z) = 2$$

Proof :(of the theorem)

*Let x, y, z three integers ≥ 2, Suppose that $\exists\ a,b,c \in \mathbb{N}^*$ such that: $gcd(a,b,c) = 1$ and $a^x + b^y = c^z$ and show that $n = \min(x,y,z) = 2$

*Suppose contrarily that: $n = \min(x,y,z) \geq 3$

Remark: by propositions 11 and 12 we can also suppose x, y, z to be prime integers such that $\gcd(x,y,z) = 1(\Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{Z}$ such that $\alpha x + \beta y + \gamma z = 1)$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$

*The proof of the theorem will be deduced from the below lemmas.

Lemma1: $\gcd(a,b,c) = 1 \Rightarrow$we can suppose $0 < a^x < b^y < c^z$ and $2a^x < c^z < 2b^y$

Proof: (of lemma1)

*We have:$a^x = c^z - b^y > 0 \Rightarrow b^y < c^z$.

*The order " $\leq$ " being total, on $\mathbb{N}$, we have:$b^y \geq a^x or\ a^x \geq b^y$

*So, we can suppose: $a^x \leq b^y$

*But $z \geq 2$ and $gcd(a,b,c) = 1 \Rightarrow a^x \neq b^y$.Indeed, if not we have:$a^x + b^y = 2b^y = c^z$ and $c \neq 0$ (so: c $\geq$ 2) $\Rightarrow$ (by the Gauss theorem, 2 being a prime integer): 2 is a divisor of c. writing $c = 2^p q$ with q an odd integer, we have:$b^y = 2^{pz-1}q^z$. Then b being an odd integer (Because: $gcd(a,b,c) = 1$ and c an even integer), we must have: $pz - 1 = 0$, which is impossible because of our hypothesis $z \geq 2$.

*So, we can suppose: $a^x < b^y$

*Finally, we have: $2b^y = b^y + b^y > c^z = a^x + b^y > a^x + a^x = 2a^x$

*In conclusion, we can suppose: $0 < a^x < b^y < c^z$ with $2a^x < c^z < 2b^y$

Now

*Working with integers: $a^x > 0 \Rightarrow a \geq 1$

First case: $a = 1$

Remark: This case is possible because $gcd(a,b,c) = 1$

*By the Catalan-Mihailescu theorem (Proposition3 given above in the § « Preliminaries »), the sole solution of the equation: $a^x + b^y = c^z$ for $a = 1$ (with: x, y, z, b, c integers>1) is :$(b,c,y,z) = (2,3,3,2)$. So, we have effectively: $n = \min(x,y,z) = \min(x,3,2) = 2$.

*Suppose, now, that we are in the second case:

Second case: $a \geq 2$

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Lemma2: We have: $a^x + b^y = c^z, \gcd(a,b,c) = 1 \Rightarrow 3 \leq a$ or $3 \leq b$
Proof: (of lemma2)
*We have: $1 < 2^x \leq a^x < b^y \Rightarrow a > 1$ and $b > 1$
*Suppose contrarily that $1 < a < 3$ and $1 < b < 3$ we have: $a = b = 2$ i.e. $2^x + 2^y = c^z$
*So, by the Gauss theorem, the prime integer 2 dividing 2^x and 2^y, 2 divides c^z and 2 divides c.
*But: 2 divides $c \Rightarrow$ 2 divides $\gcd(a,b,c) = \gcd(2,2,c) = 1$ (By the assertion (iii) of proposition1)
*This being impossible the result follows.
Lemma3: We have:
(1)$a \geq 2$ And $n = \min(x,y,z) \geq 3 \Rightarrow c^z > b^y > a^x > e^2$
(2) $a \geq 2$ And $n = \min(x,y,z) \Rightarrow c^z > b^y > a^x > e^{\frac{2}{3}n}$
(3) $a \geq 2$ And $n = \min(x,y,z) \geq 3 \Rightarrow c^z > b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
Proof: (of lemma3)
(1)*Suppose contrarily that: $a^x \leq e^2$
*We have: $a \geq 2$ and $n = \min(x,y,z) \geq 3 \Rightarrow x \geq 3$ and $a \geq 2 \Rightarrow 2 \leq a \leq e^{\frac{2}{x}} \leq e^{\frac{2}{3}} = 1.94$
*This being impossible, the result follows.
(2)*Suppose contrarily that: $a^x \leq e^{\frac{2n}{3}}$
*We have: $a \geq 2$ and $n = \min(x,y,z) \Rightarrow \frac{n}{x} \leq 1$ and $a \geq 2 \Rightarrow 2 \leq a \leq e^{\frac{2n}{3x}} \leq e^{\frac{2}{3}} = 1.94$ ...
*This being impossible, the result follows.
(3)*By lemma2, we can distinguish two cases:
First case: $a \geq 3$
*Suppose contrarily that: $a^x \leq e^{\frac{4n(n+3)}{3(2n+3)}}$
*We have: $u(t) = \frac{t+3}{2t+3} \Rightarrow u'(t) = \frac{(2t+3)-2(t+3)}{(2t+3)^2} = \frac{3-6}{(2t+3)^2} = -\frac{3}{(2t+3)^2} < 0$
*So: $n \geq 3 \Rightarrow \frac{n+3}{2n+3} \leq \frac{3+3}{2\times3+3} = \frac{6}{9} = \frac{2}{3} \Rightarrow \frac{4(n+3)}{3(2n+3)} \leq \frac{4\times2}{3\times3} = \frac{8}{9}$
*Then: $\frac{n}{x} \leq 1$ and $a \geq 3 \Rightarrow 3 \leq a \leq e^{\frac{n}{x}\left(\frac{4(n+3)}{3(2n+3)}\right)} \leq e^{\frac{8}{9}}$=2.43…
*This being impossible we have well: $a^x > e^{\frac{4n(n+3)}{3(2n+3)}}$
*So, by lemma1: $b^y > a^x \Rightarrow b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
Second case: $b \geq 3$
*Suppose contrarily that: $b^y \leq e^{\frac{4n(n+3)}{3(2n+3)}}$
*We have: $u(t) = \frac{t+3}{2t+3} \Rightarrow u'(t) = \frac{(2t+3)-2(t+3)}{(2t+3)^2} = \frac{3-6}{(2t+3)^2} = -\frac{3}{(2t+3)^2} < 0$
*So: $n \geq 3 \Rightarrow \frac{n+3}{2n+3} \leq \frac{3+3}{2\times3+3} = \frac{6}{9} = \frac{2}{3} \Rightarrow \frac{4(n+3)}{3(2n+3)} \leq \frac{4\times2}{3\times3} = \frac{8}{9}$
*Then: $\frac{n}{y} \leq 1$ and $b \geq 3 \Rightarrow 3 \leq b \leq e^{\frac{n}{y}\left(\frac{4(n+3)}{3(2n+3)}\right)} \leq e^{\frac{8}{9}}$=2.43…
*This being impossible we have well: $b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
*So: $b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
Conclusion: in any case we have: $b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
Lemma4 : (1) The function $f(t) = t^{-\frac{1}{2}}\ln(t)$ $(t > 0)$ is strictly decreasing for $t > e^2$
(2) $f(a^x) = (a^x)^{-\frac{1}{2}}\ln(a^x) > f(b^y) = (b^y)^{-\frac{1}{2}}\ln(b^y) > f(c^z) = (c^z)^{-\frac{1}{2}}\ln(c^z)$
(3) $g(t) = t^{-\frac{3}{2n}}\ln(t)$ $(t > 0)$ is strictly decreasing for $t > e^{\frac{2n}{3}}$
(4) $g(b^y) = (b^y)^{-\frac{3}{2n}}\ln(b^y) > g(c^z) = (c^z)^{-\frac{3}{2n}}\ln(c^z)$
(5) The function $h(t) = t^{-\frac{3}{2n}-1}\ln\left(et^{-\frac{3}{2n}}\right)$ $(t > 0)$ is strictly increasing for $t > e^{\frac{4n(n+3)}{3(2n+3)}}$
(6)$h(a^x) - h(a^x + b^y) = h(a^x) - h(c^z) = (a^x)^{-\frac{3}{2n}-1}\ln\left(e(a^x)^{-\frac{3}{2n}}\right) - (c^z)^{-\frac{3}{2n}-1}\ln\left(e(c^z)^{-\frac{3}{2n}}\right) < 0$
Proof : (of lemma4)
(1)*We have : $f'(t) = t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}}\ln(t) = t^{-\frac{3}{2}}\ln\left(et^{-\frac{1}{2}}\right)$

*f strictly decreasing$\Leftrightarrow \ln\left(et^{-\frac{1}{2}}\right) < 0 \Leftrightarrow et^{-\frac{1}{2}} < 1 \Leftrightarrow e < t^{\frac{1}{2}} \Leftrightarrow t > e^2$

(2) By the assertion (1) of lemma2 and the assertion (1) of lemma3, we have :

$e^2 < a^x < b^y < c^z \Rightarrow f(a^x) = (a^x)^{-\frac{1}{2}}\ln(a^x) > f(b^y) = (b^y)^{-\frac{1}{2}}\ln(b^y) > f(c^z) = (c^z)^{-\frac{1}{2}}\ln(c^z)$

(3)*We have : $g'(t) = t^{-\frac{3}{2n}-1}\ln(et^{-\frac{3}{2n}})$

* g strictly decreasing$\Leftrightarrow \ln\left(et^{-\frac{3}{2n}}\right) < 0 \Leftrightarrow et^{-\frac{3}{2n}} < 1 \Leftrightarrow e < t^{\frac{3}{2n}} \Leftrightarrow t > e^{\frac{2n}{3}}$

(4)By the assertion (2) of lemma2 and the assertion (3) of lemma3, we have :

$c^z > b^y > e^{\frac{2n}{3}} \Rightarrow g(b^y) = (b^y)^{-\frac{3}{2n}}\ln(b^y) > g(c^z) = (c^z)^{-\frac{3}{2n}}\ln(c^z)$

(5)*We have :$h'(t) = -\left(\frac{3}{2n}+1\right)t^{-\frac{3}{2n}-2}\ln\left(et^{-\frac{3}{2n}}\right) + t^{-\frac{3}{2n}-1}\left(\frac{-\frac{3}{2n}et^{-\frac{3}{2n}-1}}{et^{-\frac{3}{2n}}}\right)$

$=-t^{-\frac{3}{2n}-2}\left(\left(\frac{3}{2n}+1\right)\ln\left(et^{-\frac{3}{2n}}\right) + \frac{3}{2n}\right) = -t^{-\frac{3}{2n}-2}\left(\left(\frac{3}{2n}+1\right)\ln\left(et^{-\frac{3}{2n}}\right) + \frac{3}{2n}\ln(e)\right)$

$= -t^{-\frac{3}{2n}-2}\left(\ln\left(e^{\frac{3}{2n}+1}t^{-\frac{3}{2n}(\frac{3}{2n}+1)}\right) + \ln(e^{\frac{3}{2n}})\right) = -t^{-\frac{3}{2n}-2}\left(\ln\left(e^{\frac{3}{2n}+1}e^{\frac{3}{2n}}t^{-\frac{3}{2n}(\frac{3}{2n}+1)}\right)\right)$

$== -t^{-\frac{3}{2n}-2}\left(\ln\left(e^{\frac{3}{n}+1}t^{-\frac{3}{2n}(\frac{3}{2n}+1)}\right)\right)$

*We have :h is strictly increasing$\Leftrightarrow \ln\left(e^{\frac{3}{n}+1}t^{-\frac{3}{2n}(\frac{3}{2n}+1)}\right) < 0 \Leftrightarrow e^{\frac{3}{n}+1}t^{-\frac{3}{2n}(\frac{3}{2n}+1)} < 1$

$\Leftrightarrow e^{\frac{3}{n}+1} < t^{\frac{3}{2n}(\frac{3}{2n}+1)} \Leftrightarrow t > e^{\frac{\frac{3}{n}+1}{\frac{3}{2n}(\frac{3}{2n}+1)}} = e^{\frac{4n(n+3)}{3(2n+3)}}$

(6) By the assertion (3) of lemma2, we have :

$c^z > b^y > e^{\frac{4n(n+3)}{3(2n+3)}} \Rightarrow h(b^y) - h(c^z) = (b^y)^{-\frac{3}{2n}-1}\ln\left(e(b^y)^{-\frac{3}{2n}}\right) - (c^z)^{-\frac{3}{2n}-1}\ln\left(e(c^z)^{-\frac{3}{2n}}\right) < 0$

<u>Lemma5 :</u> We have :$\exists\theta \in]0,\frac{1}{2}[$ such that : $\frac{1}{2}(\frac{a^x}{c^z})^\theta\frac{\ln(a^x)}{\ln(c^z)} + \theta(\frac{b^y}{c^z})^\theta\frac{\ln(b^y)}{\ln(c^z)} = \frac{1}{2}$

<u>Proof :</u> (of lemma5)

*Consider, on $[0,\frac{1}{2}]$, the continuous function :

$$\varphi(t) = \frac{1}{2}(\frac{a^x}{c^z})^t\frac{\ln(a^x)}{\ln(c^z)} + t(\frac{b^y}{c^z})^t\frac{\ln(b^y)}{\ln(c^z)} - \frac{1}{2}$$

*We have :

** by lemma1 : $a^x < c^z \Rightarrow \varphi(0) = \frac{1}{2}\left(\frac{\ln(a^x)}{\ln(c^z)} - 1\right) = \frac{\ln(\frac{a^x}{c^z})}{2\ln(c^z)} < 0$

**by the assertion (2) of lemma4, we have :

$\varphi\left(\frac{1}{2}\right) = \frac{1}{2}\left(\left(\frac{a^x}{c^z}\right)^{\frac{1}{2}}\frac{\ln(a^x)}{\ln(c^z)} + \left(\frac{b^y}{c^z}\right)^{\frac{1}{2}}\frac{\ln(b^y)}{\ln(c^z)} - 1\right)$

$=\frac{1}{2}\left(\left(\frac{a^x}{c^z}\right)^{1-\frac{1}{2}}\frac{\ln(a^x)}{\ln(c^z)} + \left(\frac{b^y}{c^z}\right)^{1-\frac{1}{2}}\frac{\ln(b^y)}{\ln(c^z)} - (\frac{a^x}{c^z} + \frac{b^y}{c^z})\right)$

$=\frac{1}{2}(\frac{a^x}{c^z}\left(\left(\frac{a^x}{c^z}\right)^{-\frac{1}{2}}\frac{\ln(a^x)}{\ln(c^z)} - 1\right) + \frac{b^y}{c^z}\left(\left(\frac{b^y}{c^z}\right)^{-\frac{1}{2}}\frac{\ln(b^y)}{\ln(c^z)} - 1\right))$

$=\frac{1}{2}(\frac{a^x}{c^z}\left(\frac{(a^x)^{-\frac{1}{2}}\ln(a^x) - (c^z)^{-\frac{1}{2}}\ln(c^z)}{(c^z)^{-\frac{1}{2}}\ln(c^z)}\right) + \frac{b^y}{c^z}\left(\frac{(b^y)^{-\frac{1}{2}}\ln(b^y) - (c^z)^{-\frac{1}{2}}\ln(c^z)}{(c^z)^{-\frac{1}{2}}\ln(c^z)}\right) > 0$

*So, by the intermediate value theorem (proposition23), we have :

$\varphi(0)\varphi\left(\frac{1}{2}\right) < 0 \Rightarrow \exists\theta \in]0,\frac{1}{2}[$ Such that : $\varphi(\theta) = 0 = \frac{1}{2}(\frac{a^x}{c^z})^\theta\frac{\ln(a^x)}{\ln(c^z)} + \theta(\frac{b^y}{c^z})^\theta\frac{\ln(b^y)}{\ln(c^z)} - \frac{1}{2}$

*The result follows.

<u>Lemma6 :</u> θ given by lemma5, we have :

(1)The function $\psi(t) = 2\theta t^{-\frac{3}{2n}}\ln(t) - (t + a^x)^{-\frac{3}{2n}}\ln(t + a^x)$ is strictly decreasing for $t > e^{\frac{4n(n+3)}{3(2n+3)}}$

(2)$\forall t > \frac{4(n+3)}{3(2n+3)}$ $\psi(t) > 0$

(3) $2\theta(b^y)^{-\frac{3}{2n}}\ln(b^y) - (c^z)^{-\frac{3}{2n}}\ln(c^z) > 0$

<u>Proof :</u> (of lemma6)

(1)*We have, by lemma5 :

$\theta \in]0, \frac{1}{2}[\Rightarrow 0 < 2\theta < 1 \Rightarrow \psi'(t) = 2\theta t^{-\frac{3}{2n}-1} \ln\left(et^{-\frac{3}{2n}}\right) - (t+a^x)^{-\frac{3}{2n}-1} \ln\left(e(t+a^x)^{-\frac{3}{2n}}\right)$
$< t^{-\frac{3}{2n}-1} \ln\left(et^{-\frac{3}{2n}}\right) - (t+a^x)^{-\frac{3}{2n}-1} \ln\left(e(t+a^x)^{-\frac{3}{2n}}\right)$

*So, by the assertion (5) of lemma4 : $\psi'(t) < 0$ for $t > e^{\frac{4n(n+3)}{3(2n+3)}}$

(2) So, By the L'Hôpital rule, we have : $\forall t > e^{\frac{4n(n+3)}{3(2n+3)}}$

$\psi(t) \geq \psi(+\infty) = 2\theta \lim_{t\to+\infty} \frac{\ln(t)}{t^{\frac{3}{2n}}} - \lim_{t\to+\infty} \frac{\ln(t+a^x)}{(t+a^x)^{\frac{3}{2n}}}$

$= 2\theta \lim_{t\to+\infty} \frac{(\ln(t))'}{(t^{\frac{3}{2n}})'} - \lim_{t\to+\infty} \frac{(\ln(t+a^x))'}{((t+a^x)^{\frac{3}{2n}})'}$ (Because : $\lim_{t\to+\infty} \frac{\ln(t)}{t^{\frac{3}{2n}}} = IF\frac{+\infty}{+\infty}$ and $\lim_{t\to+\infty} \frac{\ln(t+a^x)}{(t+a^x)^{\frac{3}{2n}}} = IF\frac{+\infty}{+\infty}$)

$= 2\theta \lim_{t\to+\infty} \frac{1}{\frac{3}{2n} t \times t^{\frac{3}{2n}-1}} - \lim_{t\to+\infty} \frac{1}{\frac{3}{2n}(t+a^x)\times(t+a^x)^{\frac{3}{2n}-1}}$

$= 2\theta \lim_{t\to+\infty} \frac{1}{\frac{3}{2n} t^{\frac{3}{2n}}} - \lim_{t\to+\infty} \frac{1}{\frac{3}{2n}(t+a^x)^{\frac{3}{2n}}} = 2\theta \times 0 - 0 = 0$

(3)*By the assertion (3) of lemma3, we have : $b^y > e^{\frac{4n(n+3)}{3(2n+3)}}$
*So, by the assertion (2) of lemma6 ; we have :
$\psi(b^y) = 2\theta t^{-\frac{3}{2n}} \ln(b^y) - (a^x+b^y)^{-\frac{3}{2n}} \ln(a^x+b^y) = 2\theta (b^y)^{-\frac{3}{2n}} \ln(b^y) - (c^z)^{-\frac{3}{2n}} \ln(c^z) > 0$

Lemma7: θ given by lemma 5, we have: $n < \frac{\frac{3}{2}}{1-\theta}$

Proof: (of lemma7)

*Suppose contrarily that: $n \geq \frac{\frac{3}{2}}{1-\theta}$ i. e. $\theta \leq \frac{n-\frac{3}{2}}{n}$
*So: by lemma1, the assertion (4) of lemma4, lemma5, the assertion (3) of lemma6, and the hypothesis: $\frac{a^x}{c^z} + \frac{b^y}{c^z} = 1$, we have successively:

$0 = \frac{1}{2}\left(\frac{a^x}{c^z}\right)^\theta \frac{\ln(a^x)}{\ln(c^z)} + \theta\left(\frac{b^y}{c^z}\right)^\theta \frac{\ln(b^y)}{\ln(c^z)} - \frac{1}{2} \geq \frac{1}{2}\left(\frac{a^x}{c^z}\right)^{1-\frac{3}{2n}} \frac{\ln(a^x)}{\ln(c^z)} + \theta\left(\frac{b^y}{c^z}\right)^{1-\frac{3}{2n}} \frac{\ln(b^y)}{\ln(c^z)} - \frac{1}{2}\left(\frac{a^x+b^y}{c^z}\right)$

$= \frac{1}{2}\frac{a^x}{c^z}\left(\left(\frac{a^x}{c^z}\right)^{-\frac{3}{2n}} \frac{\ln(a^x)}{\ln(c^z)} - 1\right) + \left(\frac{b^y}{c^z}\right)\left(\theta\left(\frac{b^y}{c^z}\right)^{-\frac{3}{2n}} \frac{\ln(b^y)}{\ln(c^z)} - \frac{1}{2}\right)$

$= \frac{1}{2}\frac{a^x}{c^z}\left(\frac{(a^x)^{-\frac{3}{2n}}\ln(a^x) - (c^z)^{-\frac{3}{2n}}\ln(c^z)}{(c^z)^{-\frac{3}{2n}}\ln(c^z)}\right) + \frac{b^y}{c^z}\left(\frac{2\theta(b^y)^{-\frac{3}{2n}}\ln(b^y) - (c^z)^{-\frac{3}{2n}}\ln(c^z)}{2(c^z)^{-\frac{3}{2n}}\ln(c^z)}\right) > 0$

*So, we have obtained the assertion "0>0"
*This being impossible, the result follows.

Lemma8: (the wanted result) $n = 2$.

Proof: (of lemma8)

*The function: $u(t) = \frac{1}{1-t}$ is strictly increasing on $[0, \frac{1}{2}]$because $u'(t) = \frac{1}{(1-t)^2} > 0$.
*So, by the lemmas 5, 7 and our absurd reasoning hypothesis "$n \geq 3$", we have successively:

$$\theta \in]0, \frac{1}{2}[\Rightarrow 3 \leq n < \frac{\frac{3}{2}}{1-\theta} < \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$$

*That is, we have obtained the assertion "3<3".
*This being impossible, our hypothesis "≥ 3" is false and so we have: $n = 2$.

Conclusion: Lemma8 finishes the proof of theorem1.

Corollary: (1)$n = \min(x,y,z) > 2 \Rightarrow \forall a,b,c \in \mathbb{N}^*(a^x + b^y \neq c^z \ or \ \gcd(a,b,c) \neq 1)$
(2) i.e. $n = \min(x,y,z) > 2$ and $\exists a,b,c \in \mathbb{N}^*$ such that $a^x + b^y = c^z \Rightarrow \gcd(a,b,c) \neq 1$

Proof: (of the corollary)

(1)*The result is obtained by taking the contrapositive proposition of the proposition given in theorem1, using proposition 18.
(2)*By proposition18:
The assertion (1) $\Leftrightarrow non(n > 2) or (\forall a,b,c \in \mathbb{N}^*(a^x + b^y \neq c^z \ or \ \gcd(a,b,c) \neq 1))$
$\Rightarrow non(n > 2) or \ (\forall a,b,c \in \mathbb{N}^* a^x + b^y \neq c^z) \ or \ (\exists a,b,c \in \mathbb{N}^* \gcd(a,b,c) \neq 1)$
$\Leftrightarrow non(n > 2) or \ non(\exists a,b,c \in \mathbb{N}^* a^x + b^y = c^z) \ or \ (\exists a,b,c \in \mathbb{N}^* \gcd(a,b,c) \neq 1)$
$\Rightarrow non(n > 2 \ and \ \exists a,b,c \in \mathbb{N}^* a^x + b^y = c^z) or \ \gcd(a,b,c) \neq 1$

$\Leftrightarrow (n = \min(x, y, z) > 2) and\ (\exists a, b, c \in \mathbb{N}^*$ such that $a^x + b^y = c^z) \Rightarrow \gcd(a, b, c) \neq 1$

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