[Ghanim *et al.*, 9(2): February, 2022] ISSN 2349-0292
Impact Factor 3.802

CONFIRMATION OF THE ABC-CONJECTURE AND DEDUCTION OF SOME CONSEQUENCES

Mohammed Ghanim
* Ecole Nationale de Commerce et de Gestion
B.P 1255 Tanger Maroc

ABSTRACT

I confirm, in the present short note, the famous abc-conjecture, remained open since 1985 and described by Dorian Goldfeld in 1996 (See [43]) to be « the most important unsolved problem in Diophantine analysis », by using elementary tools of mathematics such as the intermediate value theorem, the L'Hôpital rule and the growth properties of some elementary functions. Various attempts to prove the conjecture have been made, but none are currently accepted by the main Stream mathematical community such as the very long-in 600 pages- proof [71] by the Japanese Mathematician Schinichi Mochizuki (Born in March 29, 1969)-published in 8/2012- declared- by Peter Sholze and Jacob Stix in september 2018- that it « is, in state, not receivable » (See [91]). Some important consequences, such as the Fermat last theorem, the Beal conjecture, the Roth theorem, the Fermat-Catalan conjecture, the Wieferich-Silverman theorem, the Erdos-Woods conjecture…, are deduced.

KEYWORDS: abc- conjecture; Intermediate value theorem; L'Hôpital rule; Increasing function; decreasing functions; Extremums; Fermat last theorem; The Beal conjecture; The Roth theorem; The Fermat/Catalan conjecture; The Wieferich/Silverman theorem; The Erdos/Woods conjecture.

2010 Mathematics Subject Classification: 11 A xx (Elementary Number theory)

INTRODUCTION

Definition1: We call the abc-conjecture (or the Oesterlé-Masser conjecture) the following assertion: « $\forall\epsilon > 0\exists K_\epsilon > 0$ such that$\forall$ $(a,b,c) \in A = \{(p,q,r) \in \mathbb{Z}_*^3, \gcd(p,q,r) = 1 \text{ and } p+q=r\}$, we have: $\max(|a|,|b|,|c|) \le K_\epsilon(r(abc))^{1+\epsilon}$ », where: $\gcd(p,r,q)$denotes the greatest Common divisor of the integers $|p|,|q|,|r|$ and $r(n) = \prod_{i=1}^{m(n)} p_i$ denotes the radical of n, if $|n| = \prod_{i=1}^{m(n)} {p_i}^{\alpha_i}$ is the decomposition of the integer n, according to the Fundamental Arithmetical theorem (p_i are prime integers (Recall that a positive integer p is prime if its set of divisors is $D(p) = \{1,p\}$)). An element (a,b,c) of the set A is called an abc-triple. If $\gcd(a,b,c) = 1$: we say that the integers a,b and c are coprime. Note that the name "abc-conjecture" derives from the use of letters a,b and c in the relation:

$$\text{“}\ \forall\epsilon > 0\exists K_\epsilon > 0 \text{ such that } \forall\ (a,b,c) \in \mathbb{Z}_*^3 \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \end{cases} \Rightarrow \max(|a|,|b|,|c|) \le K_\epsilon\big(r(abc)\big)^{1+\epsilon}\text{"}$$

Remark: 1) the hypothesis « $\epsilon > 0$ » is necessary in the statement of the abc-conjecture. Indeed, if $\epsilon = 0$:

* considering $x_n = 3^{2^n}, y_n = -1$ and $z_n = 3^{2^n} - 1$, we have: $\begin{cases} x_n + y_n = z_n \\ \gcd(x_n,y_n,z_n) = 1 \end{cases}$.

*But 2^{n+2} dividing z_n (for: $n \in \mathbb{N}^*$), we have:

$$** \ r(x_ny_nz_n) = r\left(3^{2^n}2^{n+2}\frac{z_n}{2^{n+2}}\right) = r(3^{2^n})r(2^{n+2})r\left(\frac{z_n}{2^{n+2}}\right) = 3.2r\left(\frac{z_n}{2^{n+2}}\right) \le 2.3.\frac{z_n}{2^{n+2}}.$$

$$**\max(|x_n|,|y_n|,|z_n|) \ge |z_n| \ge \frac{2^{n+1}r(x_ny_nz_n)}{3} = 2^{n-1}r(x_ny_nz_n).\frac{4}{3} > 2^{n-1}r(x_ny_nz_n)\ .$$

* So: (x_n,y_n,z_n) is an abc-triple exemple for which the quantity: $\frac{\max(|x_n|,|y_n|,|z_n|)}{r(x_ny_nz_n)}$, being $> 2^{n-1}$, takes arbtrairily great values.

2) The polynomial version of the abc-conjecture is the following theorem called Mason-Stothers theorem:

Theorem :(Mason-Stothers theorem) (Proved by W. Stothers in 1981 [105] and elementarily by R .C. Mason in 1984 [64]): if $n_o(d)$ denotes the number of distinct roots of the polynomial d, then for any polynomials $a(t), b(t), c(t)$; we have: $\begin{cases} \gcd(a,b,c)=1 \\ a+b=c \end{cases} \Rightarrow \max(\deg(a), \deg(b), \deg(c)) \leq n_o(abc)-1.$

3) An easy Corollary of the Mason-Stothers theorem is the below polynomial Fermat theorem:

Theorem : The equation : $(a(t))^n + (b(t))^n = (c(t))^n$, with $a(t), b(t), c(t)$ non constant polynomials, has no solution for $n \geq 3$.

History: This conjecture was first proposed by David Masser in 1985 [66] and Joseph Oesterlé in 1988[81]. It has remained open since 1985 although many attempts of eminent mathematicians. The actual state of the ABC conjecture can be summered as below:

1) **2001/Stewart and Yu's inequality**: In 2001, Stewart and Yu showed [102], [103] the weaker form of the abc-conjectures, saying that:

$$\forall\epsilon > 0 \exists K(\epsilon) > 0 \text{ such that } \forall(a,b,c) \text{ integers } \begin{cases} \mathrm{a+b=c} \\ \mathrm{gcd(a,b,c)=1} \end{cases} \Rightarrow \max(\ln(|a|), \ln(|b|), \ln(|c|)) \leq K(\epsilon)\left(r(abc)\right)^{\frac{1}{3}+\epsilon}$$

2) **2006/ABC@Home project**:"In 2006, the Mathematics Department of Leiden University in the Netherlands, together with the Dutch Kennislink science institute, launched the ABC@Home project, a grid computing system, which aims to discover additional triples a, b, c with r(abc) < c. As of May 2014, the ABC@Home had found 23.8 million triples" (See [1], [120]).

3) **2007/Lucien Szpiro proof**:" In 2007, Lucien Szpiro proposed a proof of the abc-conjecture, but it was found to be incorrect shortly afterwards" (See [120] and its references).

4) **2012/Shinichi Mochizuki proofs:** "In August 2012 Shinichi Mochizuki claimed a proof of the abc conjecture. He released a series of four preprints developing a new theory called inter-universal Teichmüller theory (IUTT) which is then applied to prove several famous conjectures in number theory, including the abc-conjecture but also Szpiro's conjecture, the hyperbolic Vojta's conjecture. The papers have not been accepted by the mathematical community as providing a proof of abc-conjecture. This is not only because of their difficulty to understand and length, but also because at least one specific point in the argument has been identified as a gap by some other experts. Though a few mathematicians have vouched for the correctness of the proof, and have attempted to communicate their understanding via workshops on IUTT, they have failed to convince the number theory community at large" (See [71], [120]).

5) **2016/Joseph Sheppard judgment**: In 2016, Joseph Sheppard wrote [93]: "we are far from proving this conjecture (The abc-conjecture). The best we can do is Stewart and Yu's 2001[103] (see also [102]) result…" [120].

6) **March-2018/Peter Scholze and Jakob Stix judgment:"** In March 2018, Peter Scholze and Jakob Stix visited Kyoto for discussions with Mochizuki. While they did not resolve the differences, they brought them into clearer focus. Scholze and Stix concluded that the gap was "so severe that … small modification will not rescue the proof strategy"; Mochizuki claimed that: "they misunderstood vital aspects of the theory and made invalid simplifications" (See [91], [120]).

7) **July-2018/Mohamed Sghiar proof**: On July 2018, Mohamed Sghiar (Burgundy university) published at "IOSR JOURNAL OF MATHEMATICS" a French paper entitled:" La preuve de la conjecture abc". In any case my approach here is completely different of that of M. Sghiar (see [92]).

ISSN 2349-0292
Impact Factor 3.802

8) **2020/Kiran Kedlaya, Edward Frenkel and Nature judgment:** "On April 3, 2020, two Japanese mathematicians announced that Mochizuki's claimed proof would be published in Publications of the Research Institute for Mathematical Sciences (RIMS), a journal of which Mochizuki is chief editor. The announcement was received with skepticism by Kiran Kedlaya and Edward Frenkel, as well as being described by Nature as: "unlikely to move many researchers over to Mochizuki's camp."" (See [120]).

The note: The present short note gives a proof of the abc-conjecture using elementary tools of mathematics and some methods developed in my papers [41], [42] confirming the Beal conjecture, published by the GJAETS . The proof is based only on the intermediate value theorem, the l'Hôpital rule and the growth properties of some elementary functions. Some important consequences are deduced such as the Fermat last theorem, the Beal conjecture, the Fermat -Catalan conjecture, the Roth theorem, the Wieferich-Silverman theorem, the Erdos-Woods conjecture…

Results: our main result is:

Theorem: (confirming the abc-conjecture).

$$\forall \epsilon > 0 \exists K_\epsilon > 0 \text{ such as } \forall a,b,c \in \mathbb{Z}^* : \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \end{cases} \Rightarrow \frac{\max(|a|,|b|,|c|)}{r(abc)^{1+\epsilon}} \leq K_\epsilon$$

Methods: I proceed by the absurd reasoning using the intermediate value theorem, the l'Hôpital rule and the growth properties of some elementary functions.

Organization of the paper: The paper is organized as follows. The §1 is an introduction giving the necessary definition and some history. The § 2 gives the ingredients of the proof. The §3 gives the proof of the abc-conjecture. The §4 gives some consequences of the abc-conjecture. The §5 gives the references of the paper for further reading.

THE INGREDIENTS OF THE PROOF OF THE abc-CONJECTURE

We will need the below definitions and results for the proofs of our main results.

Definition2: (notion of division and divisor [125]) let $y, z \in \mathbb{Z}$, we say that y divides z if: $\exists z \in \mathbb{Z}\ z = xy$. We denote by $D(z)$ the set of divisors of z.

Definition3: (notion of prime integer [126]) an integer $p \geq 2$, is called prime if its set of divisors is $D(p) = \{1, p\}$. 2 is the smallest prime integer. It is the sole even prime integer. We denote by $\mathbb{P}$ the set of prime integers.

Proposition1: (The prime integers are infinite (Euclid [31])) $\mathbb{P}$ is infinite. It is a strictly increasing sequence $(p_n)_{n\geq 1}$.

Proposition2: (the arithmetical fundamental theorem [116]) we have:

$$\forall n \in \mathbb{Z}, |n| \geq 2 \exists m(n) \in \mathbb{N}^* \exists (\alpha_i)_{1\leq i \leq m(n)} \in \mathbb{N}^* \exists (p_i)_{i \leq m(n)} \subset \mathbb{P} \text{ Such that } |n| = \prod_{i=1}^{m(n)} p_i^{\alpha_i}$$

Definition4: (the radical of an integer [121]) If $|n| = \prod_{i=1}^{m(n)} p_i^{\alpha_i} \geq 2$, (as in proposition2), for an integer $n \in \mathbb{Z}$, we call the radical of n: the positive integer: $r(n) = \prod_{i=1}^{m(n)} p_i$. It is evident, because 2 is the smallest prime integer, that the function radical is defined for $|n| \geq 2$. It is evident, also, that the radical function is an even function on $\mathbb{Z}$: i.e. $r(n) = r(-n)$.

Definition5: (The greatest common divisor [127]) for $x, y \in \mathbb{Z}$, we denote by $\gcd(x,y)$ the greatest positive common divisor of x, y.

ISSN 2349-0292
Impact Factor 3.802

Proposition3: (The Bezout theorem [117]) we have:

(i) $\gcd(x,y)=1 \Leftrightarrow \exists u,v \in \mathbb{Z}\ ux+vy=1.$
(ii) $\gcd(x,y)=1 \Leftrightarrow \forall n,m \in \mathbb{N}^*\ \ \gcd(x^n,y^m)=1.$

Definition6: (The lowest common multiple [128]) for $x,y \in \mathbb{Z}$, we denote by $\operatorname{lcm}(x,y)$ the lowest positive common multiple of x,y.

Proposition4: ([127], [128]) if $x=\prod_{i=1}^{n} p_i^{\alpha_i}$ and $y=\prod_{i=1}^{n} p_i^{\beta_i}$ are the arithmetical fundamental decompositions, then:

$$\gcd(x,y)=\prod_{i=1}^{n} p_i^{\min(\alpha_i,\beta_i)} \text{ And } \operatorname{lcm}(x,y)=\prod_{i=1}^{n} p_i^{\max(\alpha_i,\beta_i)}$$

Proposition5 :(Euler [32]-Gauss theorem [38] (See also [40], [68], [132])) we have:

The Fermat equation $a^n+b^n=c^n$ has no solutions in $\mathbb{N}^*$ for $n=3,4,5$ (i. e $\forall a,b,c \in \mathbb{N}^*$ we have: $|a^n+b^n-c^n|>0$ for:$n=3,4,5$).

Proposition 6: (Intermediate value theorem [115]) let $f:[a,b]\to\mathbb{R}$ (With: $a<b$) a continuous function. Then:

$$f(a)f(b)<0 \Rightarrow \exists c \in]a,b[\text{ such that } f(c)=0$$

Proposition9: (the equivalent versions of the abc-conjecture) the below assertions are equivalent:

(i)(**A. Baker [3], [4]**): For any positive abc-triples (a,b,c) we have :

(1) $\max(|a|,|b|,|c|) \le \frac{6}{5} r(abc)\frac{(\ln(r(abc)))^{\omega}}{\omega!}$ Where $\omega=\omega(abc)$denotes the number of distinct primes dividing abc.

(2) In particular, we have: $\max(|a|,|b|,|c|) < (r(abc))^{\frac{7}{4}}$.

Remark : Laishram and Shorey showed in [56] that for $0<\epsilon\le\frac{3}{4}$, there exists ωϵ depending only of ϵ such that when r = r (abc) ≥ $r_\epsilon=\prod_{p\le\omega_\epsilon} p$, we have c < $K_\epsilon r^{1+\epsilon}$ where $K_\epsilon=\frac{6}{5\sqrt{2\pi\max(\omega,\omega_\epsilon)}}\le\frac{6}{5\sqrt{2\pi\omega_\epsilon}}$ with ω = ω(r). Here are some values of ϵ, ω_ϵ and r_ϵ :

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6460
r_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

(ii)(**Oesterlé [81]-Masser [66]):** (a) Let $\epsilon>0$, the set OM= $\{(a,b,c)\in\mathbb{N}_*^{\,3}$ such that: $\begin{cases} a+b=c \\ \gcd(a,b,c)=1 \\ c>r(abc)^{1+\epsilon}\end{cases}\}$ is finite.

(iii) $\forall\epsilon>0\exists K(\epsilon)>0$ Such that for any positive abc-triple (a,b,c) we have: $r(abc)>K(\epsilon)c^{1-\epsilon}$.

(iv)(**The quality version of the abc-conjecture**):

$$\forall \epsilon > 0 \exists \text{ only finitely many positive abc} - \text{triples } (a,b,c) \text{such that } q(a,b,c) = \frac{\ln(c)}{\ln(r(abc))} > 1 + \epsilon$$

The quantity $q(a,b,c)$ is called the quality of (a, b, c).

(v)(**Granville-Tucker [45]**) The abc-conjecture says the lim sup of the set of all qualities (defined above) is 1.

Remark: This implies the much weaker assertion that there is a finite upper bound for qualities. The conjecture that 2 is such an upper bound is sufficient for giving a very short proof of Fermat's Last Theorem.

Proposition10: (Circular functions [129]) recall that:

(i) $\cos(0) = 1$ (ii) $\sin(0) = 0$(iii) $\tan(t) = \frac{\sin(t)}{\cos(t)}$.

(iv) The function:$t \to \sin(t)$ is strictly increasing on $[0,\frac{\pi}{2}]$ with $(\sin(t))' = \cos(t) \geq 0$ on $[0,\frac{\pi}{2}]$.

(v) The function:$t \to cos(t)$ is strictly decreasing on $[0,\frac{\pi}{2}]$ with $(\cos(t))' = -\sin(t) \leq 0$ on $[0,\pi]$.

(vi) The function: $t \to \tan(t)$ is strictly increasing on $[0,\frac{\pi}{2}]$ with $(\tan(t))' = 1 + (\tan(t))^2$=$\frac{1}{(\cos(t))^2}$, so has a reciprocal function denoted "arctan": $[0, +\infty[\to \left[0,\frac{\pi}{2}\right]$.

Proposition11: (the contrapositive proposition [118]) (i) Let (P), (Q) two mathematical propositions. Noting by non (P) the negation of the proposition P (for example non $(\exists) = \forall$, non $(\leq) = >$), we call the contrapositive of the proposition "P$\Rightarrow Q$" (equivalent to the proposition "non (P) or Q") the proposition "non (Q)$\Rightarrow non\ (P)$.

(ii)An implication and its contrapositive are equivalent i.e. we have: $(P \Rightarrow Q) \Leftrightarrow \big(non(Q) \Rightarrow non(P)\big)$.

(iii) Recall that: For any propositions (P) and (Q): (($P \Rightarrow Q$) true and P true)$\Rightarrow Q$ true.

(iv)The contradiction principle: for a proposition (P): $(P)$and $non(P)$ cannot be simultaneously true. This is the base of the absurd reasoning.

Proposition12: (Catalan-Mihailescu theorem [69], [70]) the sole solution of the Diophantine equation $1 + b^y = c^z$ is $(b,c,y,z) = (2,3,3,2)$.

Proposition 13: (The L'Hôpital rule [119]) we have:

(i) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$ (a can be infinite) the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called to be an indeterminate form (IF)$\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively.

(ii) If f, g are differentiable on an interval $]a,b[$ except perhaps in a point $c \in]a,b[$, if $\lim_{x\to c}\frac{f(x)}{g(x)}$ is the IF $\frac{0}{0}$ and if $\forall x \neq c, g'(x) \neq 0$, then: $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ when the limits have a sense.

(iii) If f', g' satisfies the same conditions as f and g the process is repeated.

(iv)The result remain true in the case where $\lim_{x\to c}\frac{f(x)}{g(x)}$ is the IF $\frac{\infty}{\infty}$.

Proposition 14: (The Bolzano-Weierstrass theorem [130]): any bounded sequence $(x_n)_n \subset]a,b[$ has a subsequence, denoted (for convenience) also by$(x_n)_n$, converging to $x = \lim_{n\to+\infty} x_n \in [a,b]$.

Definition 5: (The integer part [123]) the integer part of a real number x is the single integer number $E(x)$ such that:$E(x) \leq x < E(x)+1$ i.e.: $0 \leq x - E(x) < 1$.

THE POROOF OF THE abc-CONJECTURE

Theorem1: (Proving the abc-conjecture) we have:

$$\forall \epsilon > 0 \exists K_\epsilon > 0 \text{such that } \forall a,b,c \in \mathbb{Z}^* \text{ such that} : \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \end{cases} \Rightarrow \frac{\max(|a|,|b|,|c|)}{r(abc)^{1+\epsilon}} \leq K_\epsilon$$

Proof :(of the theorem1)

*The proof of the theorem will be deduced from the below lemmas.

*Proceed by the absurd reasoning and suppose contrarily that:

$$\exists \epsilon > 0 \forall K > 0 \ \exists a,b,c \in \mathbb{Z}^* \quad \text{Such that:} \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \\ \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} < \frac{1}{K} \end{cases}$$

Notation: *In particular: $\exists \epsilon > 0$ such that for $K = 10, \exists a,b,c \in \mathbb{Z}^*$ satisfying: $\begin{cases} a+b=c \\ \gcd(a,b,c)=1 \\ \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} < \frac{1}{10} \end{cases}$.

*putting: $n = n(a,b,c) = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)}$.: we have, by the absurd hypothesis: $0 < n < \frac{1}{10} < 1$.

Lemma1: We can suppose $a,b,c \in \mathbb{N}^*$.

Proof: (of lemma1)

*Let $a,b,c \in \mathbb{Z}^*$ such that:$c = a+b$.

*The possible signs of these integers are elements of the set: $S = \{-1,+1\} \times \{-1,+1\} \times \{-1,+1\}$.

*We have $S = \{(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1),(1,-1,-1),(1,-1,1),(1,1,-1),(1,1,1)\}$.

First case: The case $(\text{sign}(a),\text{sign}(b),\text{sign}(c)) = (-1,-1,-1)$.

We can replace: (a,b,c) by $(-a,-b,-c)$ which are all positive and we can work with relation: $-a-b=-c$.

Second case: The case $(\text{sign}(a),\text{sign}(b),\text{sign}(c)) = (-1,-1,1)$.

*We have: $0 < c = b+a < 0$.

*So this impossible case cannot occur.

Third case: The case $(\text{sign}(a),\text{sign}(b),\text{sign}(c)) = (-1,1,-1)$.

*We have: $-a'+b=-c', a'=-a>0, b>0, c'=-c>0$.

*We have:$a' - b = c'$ or $a' = b + c'$.

* We can replace (a, b, c) by $(-c, b, -a)$ which are all positive and we can work with the relation: $-a = b - c$.

Fourth case The case (sign (a), sign(b), sign(c)) $= (-1,1,1)$.

*We have: $-a' + b = c$, $b > 0, a' = -a > 0$ and $c > 0$.

*We have:$b = a' + c$.

*So we can replace: (a, b, c) by $(-a, c, b)$, which are all positive, and we can work with the relation $b = -a + c$.

Fifth case: The case (sign (a), sign(b), sign(c)) $= (1, -1, -1)$.

*We have: $a - b' = -c'$ with $a > 0, b' = -b > 0, c' = -c > 0$.

*We have: $a = b' + c'$.

* We can replace (a, b, c) by $(-b, -c, a)$, which are all positive, and we can work with the relation:$a = -b - c$.

Sixth case: The case (sign (a), sign(b), sign(c)) $= (1, -1, 1)$.

*We have: $a - b' = c$ with $a > 0, b' = -b > 0$, $c > 0$.

*We have: $a = b' + c$.

*So, we can replace (a, b, c) by $(-b, c, a)$ which are all positive and work with the relation:$a = -b + c$.

Seventh case: The case (sign (a), sign(b), sign(c)) $= (1,1, -1)$.

*We have: $0 < a + b = -c' < 0$ with $a > 0, c' = -c > 0$, $b > 0$.

*So, this impossible case cannot occur.

Conclusion: So we can work with the last eighth case: $a + b = c$, with: $a > 0, b > 0, c > 0$.

Lemma2: $\begin{cases} a + b = c \\ n = n(a, b, c) = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} < \frac{1}{10} \\ \gcd(a, b, c) = 1 \\ abc \neq 0 \end{cases}$ $\Rightarrow$we can suppose $1 \leq a < b < c$.

Proof: (of lemma2)

*We have:$a = c - b > 0 \Rightarrow a \geq 1$ and $b < c$.

*The order " $\leq$ " being total on$\mathbb{N}$, we have:$b \geq a$ or $a \geq b$.

*So, we can suppose: $a \leq b$.

*But: $gcd(a, b, c) = 1 \Rightarrow a \neq b$.Indeed, if not we have:$a + b = 2a = c$, and so, we have:

$$\gcd(a, b, c) = \gcd(a, a, 2a) = a = 1$$

*That is: $a = b = 1$ and $c = 2$ (We have well: $\gcd(a, b, c) = 1$.

ISSN 2349-0292
Impact Factor 3.802

*So: $\frac{1}{10} > \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} = \frac{(r(1.1.2))^{1+\epsilon}}{\max(1,1,2)} = \frac{2^{1+\epsilon}}{2} = 2^{\epsilon} > 1$ (because: $\epsilon > 0$).

*This being impossible, we can suppose: $a < b$.

Lemma3: We have: $\begin{cases} a+b=c \\ n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} < \frac{1}{10} \\ \gcd(a,b,c)=1 \end{cases} \Rightarrow a \geq 3$ or $b \geq a+3$.

Proof: (of lemma3)

*Suppose contrarily that: $b < a+3$ and $a < 3$.

*$\begin{cases} c = a+b \\ a < 3 \\ b < a+3 \end{cases} \Rightarrow c = a+b < a+6 < 3+6 = 9.$

*So: $\begin{cases} c < 9 \\ (r(abc))^{1+\epsilon} \geq 1 \end{cases} \Rightarrow \frac{1}{10} > n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} = \frac{(r(abc))^{1+\epsilon}}{c} \geq \frac{1}{c} > \frac{1}{9}$.

*Obtaining the impossible assertion "$\frac{1}{9} < n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} < \frac{1}{10}$", we have well: $a \geq 3$ or $b \geq a+3$.

Remark: (i) In the case $1 \leq a < 3$, we have: $c+2 > c > b \geq a+3 > a+2 > 3$ with: $\frac{a+2}{c+2} + \frac{b}{c+2} = 1$ (because:$a+b=c$). So, we can work with the triple $(a+2, b, c+2)$.

(ii) The condition $b \geq a+3$ is possible because of the inequality "$b > a$", assured by lemma2 (under the hypothesis: $\gcd(a,b,c) = 1$).

Lemma4: For $k \geq 2$ and $n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)}$ we have:

(1)The function $u(t) = t^{-\frac{\pi}{4}} (\ln(t))^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} (t \geq 1)$ is strictly decreasing for $t \geq e^{n(\frac{\pi}{4}+\frac{1}{k})}$.

(2) (i) if <u>: $a \geq 3$,</u> we have: $c > b > a \geq 3 > e^{n(\frac{\pi}{4}+\frac{1}{k})} \Rightarrow u(a) > u(b) > u(c)$.

(ii) if<u>: $a < 3$</u>, we have:$c+2 > b \geq a+2 \geq 3 > e^{n(\frac{\pi}{4}+\frac{1}{k})} \Rightarrow u(a+2) > u(b) > u(c+2)$.

(3)If $S(k) = \lambda(k) or\ \theta(k) \in]1-\frac{\pi}{4}, 1[$ are as defined in the lemma 6 below, the function $v(t, S(k)) = t^{-n\tan(1-S(k))} (\ln(t))^{n\frac{\pi}{4}(1-S(k)+\frac{1}{k})} (t \geq 1)$ is strictly decreasing for $t \geq e^{\frac{\frac{\pi}{4}(1-S(k)+\frac{1}{k})}{\tan(1-S(k))}}$.

(4)$\exists p \geq 2$ Such that: $3 \geq e^{\frac{\frac{\pi}{4}(1-S(p)+\frac{1}{p})}{\tan(1-S(p))}}$.

(5)$p \geq 2$, being the number given by the assertion (4) of lemma4, we have:

(i) ***If $a \geq 3$**: $c > b > a \geq 3 \geq e^{\frac{\frac{\pi}{4}(1-\lambda(p)+\frac{1}{p})}{\tan(1-\lambda(p))}} \Rightarrow v(a, \lambda(p)) > v(b, \lambda(p)) > v(c, \lambda(p))$.

(ii)***If $a < 3$**: $c+2 > b > a+2 \geq 3 \geq e^{\frac{\frac{\pi}{4}(1-\theta(p)+\frac{1}{p})}{\tan(1-\theta(p))}} \Rightarrow v(a+2, \theta(p)) > v(b, \theta(p)) > v(c+2, \theta(p))$.

 ISSN 2349-0292
Impact Factor 3.802

Proof: (of lemma4)

(1) (2) (i), (ii) the result follows immediately from the assertion (1) of lemma4 because:

*We have: $u'(t) = t^{-\frac{\pi}{4}-1}\,(\ln(t))^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})-1}\ln\left(t^{-\frac{\pi}{4}}e^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right)$.

*We have:u Strictly decreasing$\Leftrightarrow \ln\left(t^{-\frac{\pi}{4}}e^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right) < 0 \Leftrightarrow t^{-\frac{\pi}{4}}e^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} < 1$.

$\Leftrightarrow t^{\frac{\pi}{4}} > e^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} \Leftrightarrow t > e^{n(\frac{\pi}{4}+\frac{1}{k})}$.

$\begin{cases} k \geq 2 \\ 0 < n < \frac{1}{10} \end{cases} \Rightarrow \frac{1}{10}\left(\frac{\pi}{4}+\frac{1}{2}\right) > n\left(\frac{\pi}{4}+\frac{1}{k}\right) > 0 \Rightarrow 3 > e^{\frac{1}{10}\left(\frac{\pi}{4}+\frac{1}{2}\right)} = 1.1371\ldots > e^{n\left(\frac{\pi}{4}+\frac{1}{k}\right)}$.

(3)*We have:

$v'(t, S(k)) = t^{-n\tan(1-S(k))-1}(\ln(t))^{n\frac{\pi}{4}(1-S(k)+\frac{1}{k})-1}\ln\left(t^{-n\tan(1-S(k))}e^{n\frac{\pi}{4}\left(1-S(k)+\frac{1}{k}\right)}\right)$.

*We have:$t \to v(t, S(k))$ Strictly decreasing$\Leftrightarrow \ln\left(t^{-n(\tan(1-S(k)))}e^{n\frac{\pi}{4}(1-S(k)+\frac{1}{k})}\right) < 0$.

$\Leftrightarrow t^{-n(\tan(1-S(k)))}e^{n\frac{\pi}{4}(1-S(k)+\frac{1}{k})} < 1 \Leftrightarrow t^{n(\tan(1-S(k)))} > e^{n\frac{\pi}{4}(1-S(k)+\frac{1}{k})} \Leftrightarrow t > e^{\frac{\frac{\pi}{4}\left(1-S(k)+\frac{1}{k}\right)}{\tan(1-S(k))}}$.

(4)*Suppose contrarily that: $\forall k \geq 2$: $e^{\frac{\frac{\pi}{4}\left(1-S(k)+\frac{1}{k}\right)}{\tan(1-S(k))}} \geq 3$.

*Tending: $k \to +\infty$, we have, by the Bolzano-Weierstrass theorem:

$\lim_{k\to+\infty} S(k) = S$ and $\frac{(1-S)}{\tan(1-S)} \leq 1 \Rightarrow e^{\frac{\pi}{4}} = 2.1932\ldots \geq e^{\frac{\frac{\pi}{4}(1-S)}{\tan(1-S)}} \geq 3$.

*This being impossible, the result follows.

(5)The result follows, from the assertion (4) of lemma4.

Lemma5: We have:

(1) $\lim_{t\to 0}\frac{t}{\tan(t)} = 1$.

(2)$\forall t \in \left[0, \frac{\pi}{2}\right]$ $\varphi(t) = \tan(t) - t \geq 0$.

(3)$\forall t \in \left[0, \frac{\pi}{4}\right]$ $\tau(t) = t - \frac{\pi}{4}\tan(t) \geq 0$.

Proof: (of lemma 5)

(1)By the L'Hôpital rule, we have: $\lim_{t\to 0}\frac{t}{\tan(t)} = FI\,\frac{0}{0} = \lim_{t\to 0}\frac{t'}{(\tan(t))'} = \lim_{t\to 0}\frac{1}{1+(\tan(t))^2} = \frac{1}{1+(\tan(0))^2} = 1$.

(2)We have: $\varphi'(t) = 1 + (\tan(t))^2 - 1 = (\tan(t))^2 \geq 0$ $\forall t \in [0, \frac{\pi}{2}] \Rightarrow \varphi$ increasing on $\left[0, \frac{\pi}{2}\right] \Rightarrow \forall t \in [0, \frac{\pi}{2}]\varphi(t) = \tan(t) - t \geq \varphi(0) = 0$.

(3)*We have: $\tau'(t) = 1 - \frac{\pi}{4}(1 + (\tan(t))^2)$.

*$\tau'(t) = 0, t \in \left[0, \frac{\pi}{4}\right] \Leftrightarrow t = \alpha = \arctan\left(\sqrt{\frac{4}{\pi} - 1}\right) \in \left[0, \frac{\pi}{4}\right]$.

* On $[0, \alpha]$: τ is increasing and on $[\alpha, \frac{\pi}{4}]$: τ is decreasing.

*So:

** $\forall t \in [0, \alpha]\ \ \tau(t) = t - \frac{\pi}{4}\tan(t) \geq \tau(0) = 0$.

**$\forall t \in \left[\alpha, \frac{\pi}{4}\right]\ \ \tau(t) = t - \frac{\pi}{4}\tan(t) \geq \tau\left(\frac{\pi}{4}\right) = 0$.

*The result follows because: $\left[0, \frac{\pi}{4}\right] = [0, \alpha] \cup \left[\alpha, \frac{\pi}{4}\right]$.

Lemma6: We have:

(1)(i) **if: $a \geq 3$** , we have: $\forall k \geq 2 \exists \lambda(k) \in]1 - \frac{\pi}{4}, 1[$ Such that:

$$\left(\frac{a}{c}\right)^{\lambda(k)} \left(\frac{\ln(a)}{\ln(c)}\right)^{n\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} + \left(\frac{b}{c}\right)^{\lambda(p)} \left(\frac{\ln(b)}{\ln(c)}\right)^{n\frac{\pi}{4}(1-\lambda(k)+\frac{1}{k})} = 1$$

(ii) $\lim_{k\to+\infty} \lambda(k) = \lambda \in \left[1 - \frac{\pi}{4}, 1\right]$.

(2)(i) **if: $a < 3$** , we have: $\forall k \geq 1\ \exists \theta(k) \in]1 - \frac{\pi}{4}, 1[$ Such that:

$$\left(\frac{a+2}{c+2}\right)^{\theta(k)} \left(\frac{\ln(a+2)}{\ln(c+2)}\right)^{n\frac{\pi}{4}(1-\theta(k)+\frac{1}{k})} + \left(\frac{b}{c+2}\right)^{\theta(k)} \left(\frac{\ln(b)}{\ln(c+2)}\right)^{n\frac{\pi}{4}(1-\theta(k)+\frac{1}{k})} = 1$$

(ii) $\lim_{k\to+\infty} \theta(k) = \theta \in \left[1 - \frac{\pi}{4}, 1\right]$.

Proof: (of lemma6)

(1)(i)*suppose: $a \geq 3$ and consider, for $k \geq 2$, the continuous function defined on $[1-\frac{\pi}{4}, 1]$ by:

$$f(t) = \left(\frac{a}{c}\right)^t \left(\frac{\ln(a)}{\ln(c)}\right)^{n\frac{\pi}{4}(1-t+\frac{1}{k})} + \left(\frac{b}{c}\right)^t \left(\frac{\ln(b)}{\ln(c)}\right)^{n\frac{\pi}{4}(1-t+\frac{1}{k})} - 1$$

*By the assertion (2) (i) of lemma4, we have:

**$3 \leq a < b < c \Rightarrow f\left(1 - \frac{\pi}{4}\right) = \left(\frac{a}{c}\right)^{1-\frac{\pi}{4}} \left(\frac{\ln(a)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} + \left(\frac{b}{c}\right)^{1-\frac{\pi}{4}} \left(\frac{\ln(b)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} - 1$

$= \left(\frac{a}{c}\right)^{1-\frac{\pi}{4}} \left(\frac{\ln(a)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} + \left(\frac{b}{c}\right)^{1-\frac{\pi}{4}} \left(\frac{\ln(b)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} - \left(\frac{a}{c} + \frac{b}{c}\right)$

$= \frac{a}{c}\left(\left(\frac{a}{c}\right)^{-\frac{\pi}{4}} \left(\frac{\ln(a)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} - 1\right) + \frac{b}{c}\left(\left(\frac{b}{c}\right)^{-\frac{\pi}{4}} \left(\frac{\ln(b)}{\ln(c)}\right)^{n\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} - 1\right) > 0.$

** We have:

$$\begin{cases} 1 < \ln(3) \leq \ln(a) < \ln(b) < \ln(c) \\ \frac{\mathrm{n}}{\mathrm{k}} > 0 \end{cases} \Rightarrow f(1) = \frac{a}{c}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} + \frac{b}{c}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} - 1$$

$$= \frac{a}{c}\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} + \frac{b}{c}\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} - \left(\frac{a}{c} + \frac{b}{c}\right)$$

$$= \frac{a}{c}\left(\left(\frac{\ln(a)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} - 1\right) + \frac{b}{c}\left(\left(\frac{\ln(b)}{\ln(c)}\right)^{\frac{\mathrm{n}\pi}{4\mathrm{k}}} - 1\right) < 0.$$

*So, by the intermediate value theorem, we have:

$$f\left(1 - \frac{\pi}{4}\right) f(1) < 0 \Rightarrow \forall k \geq 2 \exists \lambda(k) \in]1 - \frac{\pi}{4}, 1[\text{ such that: } \quad f(\lambda(\mathrm{k})) = 0$$

(ii) The result follows by application of the Bolzano-Weierstrass theorem to the bounded sequence:$(\lambda(k))_{k\geq 2} \subset]1 - \frac{\pi}{4}, 1[$.

(2) (i) If:$a < 3$ (so: $b \geq a + 3 > a + 2 \geq 3$), applying the intermediate value theorem, (with use of the relation $\frac{a+2}{c+2} + \frac{b}{c+2} = 1$), to the continuous function defined on $[1 - \frac{\pi}{4}, 1]$, by:

$$g(t) = \left(\frac{a+2}{c+2}\right)^{\mathrm{t}} \left(\frac{\ln(a+2)}{\ln(c+2)}\right)^{\mathrm{n}\frac{\pi}{4}(1-\mathrm{t}+\frac{1}{\mathrm{k}})} + \left(\frac{b}{c+2}\right)^{\mathrm{t}} \left(\frac{\ln(b)}{\ln(c+2)}\right)^{\mathrm{n}\frac{\pi}{4}(1-\mathrm{t}+\frac{1}{\mathrm{k}})} - 1$$

we obtain (exactly such as in the assertion (1) of lemma6) the result:

$$\forall k \geq 2 \; \exists \theta(k) \in]1 - \frac{\pi}{4}, 1[\text{ Such that: } g(\theta(k)) = 0$$

(ii) The result follows by application of the Bolzano-Weierstrass theorem (See proposition 14) to the bounded sequence:$(\theta(k))_{k\geq 2} \subset]1 - \frac{\pi}{4}, 1[$.

Lemma7: For $n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)}$ and p given by the assertion (4)of lemma4 and $\lambda(p), \theta(p)$ given by lemma 6, we have:

(1) $a \geq 3 \Rightarrow \; n \geq \frac{1-\lambda(p)}{\tan(1-\lambda(p))}$.

(2) $a < 3 \Rightarrow \; n \geq \frac{1-\theta(p)}{\tan(1-\theta(p))}$.

Proof: (of lemma7)

(1)*If $a \geq 3$:

*Suppose contrarily that: $n < \frac{1-\lambda(p)}{\tan(1-\lambda(p))}$.

*We have: $\lambda(p) < 1 - n\tan(1 - \lambda(p))$.

*So, by the assertion (1) (i) of lemma6 and the assertion (5) (i) of lemma 4, we have successively:

$$0 = \left(\frac{a}{c}\right)^{\lambda(\mathrm{p})} \left(\frac{\ln(a)}{\ln(c)}\right)^{\mathrm{n}\frac{\pi}{4}(1-\lambda(\mathrm{p})+\frac{1}{\mathrm{p}})} + \left(\frac{a}{c}\right)^{\lambda(\mathrm{p})} \left(\frac{\ln(a)}{\ln(c)}\right)^{\mathrm{n}\frac{\pi}{4}(1-\lambda(\mathrm{p})+\frac{1}{\mathrm{p}})} - 1.$$

$>(\frac{a}{c})^{1-\text{ntan}(1-\lambda(\text{p}))}\ (\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} + (\frac{b}{c})^{1-\text{ntan}(1-\lambda(\text{p}))}\ (\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} - 1.$

$=(\frac{a}{c})^{1-\text{ntan}(1-\lambda(\text{p}))}\ (\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} + (\frac{b}{c})^{1-\text{ntan}(1-\lambda(\text{p}))}\ (\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} - (\frac{a}{c}+\frac{b}{c}).$

$=\frac{a}{c}((\frac{a}{c})^{-\text{ntan}(1-\lambda(\text{p}))}\ (\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} - 1) + \frac{b}{c}((\frac{b}{c})^{-\text{ntan}(1-\lambda(\text{p}))}(\frac{\ln(a)}{\ln(c)})^{\text{n}\frac{\pi}{4}(1-\lambda(\text{p})+\frac{1}{\text{p}})} - 1) > 0.$

*The obtained assertion “0<0” being impossible, we have:: $n \geq \frac{1-\lambda(p)}{\tan(1-\lambda(p))}$.

(2)By analogy, if $a < 3$, the result is obtained exactly as in the assertion (1) of lemma7 with use of the assertion (2) (i) of lemma6 and the assertion (5) (ii) of lemma4.

RETURN TO THE PROOF OF THE THEOREM1:

First case: if $a \geq 3$

*By the assertion the assertion (3) of lemma5 and the assertion (1) of lemma7, we have:

$$0 < 1-\lambda(p) < \frac{\pi}{4} \Rightarrow \ \frac{1}{10} = 0.1 \geq \ n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} \geq \frac{1-\lambda(p)}{\tan(1-\lambda(p))} \geq \frac{\pi}{4} = 0.7853..$$

*This being impossible the first case cannot occur.

Second case: if $a < 3$

* By the assertion the assertion (3) of lemma5 and the assertion (2) of lemma7, we have:

$$0 < 1-\theta(p) < \frac{\pi}{4} \Rightarrow \ \frac{1}{10} = 0.1 \geq \ n = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} \geq \frac{1-\theta(p)}{\tan(1-\theta(p))} \geq \frac{\pi}{4} = 0.7853 \ldots$$

* This being impossible the second case cannot occur.

Conclusion: The two possible cases “$a \geq 3$" and “$a < 3$" of lemma 7 couldn’t both occur, the wanted contradiction is reached. So our starting absurd hypothesis:

$$(P):\text{” } \exists\epsilon > 0\ \exists a,b,c \in \mathbb{Z}^* \ \text{ such that: } \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \\ n = n(a,b,c) = \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} \leq \frac{1}{10} \end{cases} \text{" is not true.}$$

*So its negation $non(P)$ is true.

*But the assertion (Q) "$\exists\epsilon > 0\ \forall K > 0 \exists a,b,c \in \mathbb{Z}^*$ such that: $\begin{cases} a+b=c \\ \gcd(a,b,c)=1 \\ \frac{(r(abc))^{1+\epsilon}}{\max(|a|,|b|,|c|)} \leq \frac{1}{K} \end{cases}$" implies the assertion (P).

*So, by contraposition, we have: $non(P) \Rightarrow non(Q)$.

*So: $non\ (P)$ being true: $non(Q)$ is true.

*But: $non(Q)$ is: « $\forall\epsilon > 0 \exists K_\epsilon > 0 \forall a,b,c \in \mathbb{Z}^* \begin{cases} a+b=c \\ \gcd(a,b,c)=1 \end{cases} \Rightarrow \max(|a|,|b|,|c|) < K_\epsilon\ (r(abc))^{1+\epsilon}$».

*So: the abc conjecture is true.

Conclusion: This finishes the proof of the theorem1.

SOME CONSQUENCES OF THE ABC CONJECTURE:

1. THE FERMAT LAST THEOREM:

Corollary 1: (The Fermat last theorem) (See [40], [132]) the abc-conjecture implies the Fermat's Last Theorem. That is: $\exists(a,b,c) \in \mathbb{N}^3$ such that: $\begin{cases} abc \neq 0 \\ a^m + b^m = c^m \\ \gcd(a,b,c) = 1 \\ m \geq 2 \end{cases} \Rightarrow m = 2.$

Proof: (of corollary 1)

*Let $m \in \mathbb{N}^*$ with $m \geq 2$ and $(a,b,c) \in \mathbb{N}_*^3$ such that: $a^m + b^m = c^m$ and gcd (a,b,c) =1.

*Show that:$m = 2$.

*we can suppose: $a < b < c$ and $c \geq 3$.

*By the Baker abc-conjecture version, for any abc-triple (a,b,c): we have: $c < (r(abc))^2$.

*So $c^m < (r(a^m b^m c^m))^2 = (r(abc))^2 < (abc)^2 < c^6$, hence: $2 \leq m \leq 5$.

*Finally−because the Diophantine equation $a^m + b^m = c^m$ has not non trivial solutions for:$m = 3{,}4{,}5$ − we have well: $m = 2$.

2. THE BEAL CONJECTURE:

Corollary 2: (The Beal conjecture) (see [41], [42]) the abc-conjecture implies the Beal conjecture. That is: $\exists(a,b,c) \in \mathbb{N}^3$ such that: $\begin{cases} abc \neq 0 \\ a^x + b^y = c^z \\ \gcd(a,b,c) = 1 \\ x,y,z \geq 2 \end{cases} \Rightarrow m = \min(x,y,z) = 2.$

Proof: (of corollary 2)

First Method: By M. Ghanim

*Let $x,y,z \in \mathbb{N}^* \geq 2$, $(a,b,c) \in \mathbb{N}_*^3$ such that: $a^x + b^y = c^z$ and gcd (a,b,c) =1.

* Prove that $\mathrm{m} = \min(x,y,z) = 2$.

*We can suppose: $a^x < b^y < c^z$.

*For $a = 1$: the Mihailescu theorem assures that the single solution of the Diophantine equation $1 + b^y = c^z$is $(b,y,c,z) = (2{,}3{,}3{,}2)$.

*So we can suppose $a \geq 2$.

Claim1: We have: $\frac{4}{7} < \frac{1}{\mathrm{x}} + \frac{1}{\mathrm{y}} + \frac{1}{\mathrm{z}}$.

Proof: (of claim1)

*Applying the Baker Version of the abc-conjecture, we have:

$$c^z < (r(abc))^{\frac{7}{4}} < (abc)^{\frac{7}{4}} = (a^x)^{\frac{7}{4x}}(b^y)^{\frac{7}{4y}}(c^z)^{\frac{7}{4z}} < c^{\frac{7}{4}z(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})}$$

*That is: $\frac{4}{7} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

Claim2: We have: $m = \min(x, y, z) \leq 5$.

Proof: (of claim2)

*By claim1, we have: $\frac{4}{7} < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \Rightarrow \frac{4}{7} \leq \frac{3}{\min(x,\ y,\ z)}$.

*That is: $2 \leq m = \min(x, y, z) < 3(\frac{7}{4}) = \frac{21}{4} = 5.25$ i.e. $2 \leq \min(x, y, z) \leq 5$.

Claim3: We have necessarily $m = \min(x, y, z) = 2$.

Proof: (of claim3)

Indeed, proceed by the absurd reasoning and suppose that: $a^x + b^y = c^z$ with $abc \neq 0$ and $\gcd(a, b, c) = 1$,

where: $x = m + \alpha, y = m + \beta, z = m + \gamma$ with $\alpha\beta\gamma = 0$ and $m = \min(x, y, z) = 3$ or 4 or 5.

Under-claim1: (1)$\forall k \geq 1 \exists \delta(k) \in]0,2[$ such that:

$$a^{x\delta(k)}(\ln\left(a^{\alpha+\frac{1}{k}}\right))^{\delta(k)(2-\delta(k))} + b^{y\delta(k)}(\ln\left(b^{\beta+\frac{1}{k}}\right))^{\delta(k)(2-\delta(k))} = c^{z\delta(k)}(\ln\left(c^{\gamma+\frac{1}{k}}\right))^{\theta(k)(2-\delta(k))}$$

(2) $\lim_{k\to+\infty} \delta(k) = \delta \in [0,2]$.

(3)$a^{x\delta}(\ln(a^\alpha))^{\delta(2-\delta)} + b^{y\delta}(\ln(b^\beta))^{\delta(2-\delta)} = c^{z\delta}(\ln(c^\gamma))^{\delta(2-\delta)}$.

Proof: (of under-claim1)

(1)* Consider, on [0, 2] for $k \geq 1$, the continuous function:

$$h_k(t) = a^{xt}(\ln\left(a^{\alpha+\frac{1}{k}}\right))^{t(2-t)} + b^{yt}(\ln\left(b^{\beta+\frac{1}{k}}\right))^{t(2-t)} - c^{zt}(\ln\left(c^{\gamma+\frac{1}{k}}\right))^{t(2-t)}$$

*We have:

**$h_k(0) = 1 + 1 - 1 = 1 > 0$.

**$\begin{cases} 4 \leq a^x < b^y < c^z \\ \frac{a^x}{c^z} + \frac{b^y}{c^z} = 1 \end{cases} \Rightarrow h_k(2) = a^{2x} + b^{2y} - c^{2z} = c^{2z}\left(\left(\frac{a^x}{c^z}\right)^2 + \left(\frac{b^y}{c^z}\right)^2 - 1\right) < c^{2z}\left(\frac{a^x}{c^z} + \frac{b^y}{c^z} - 1\right) = 0.$

*So, by the intermediate value theorem, $\exists \delta(k) \in]0,2[$ such that: $h_k(\delta(k)) = 0$.

*The result follows.

(2)The result follows by applying the Bolzano-Weierstrass theorem to the bounded sequence: $(\delta(k))_{k\geq1} \subset]0,2[$.

(3)The result follows by tending: $k \to +\infty$ in relation (1) of claim3 with use of the assertion (2) of claim3.

Under-claim2: We have: $\alpha = \beta = \gamma = 0$.

Proof: (of the under-claim2)

Because: $\alpha\beta\gamma = 0$, we have: $\alpha = 0$ *or* $\beta = 0$ *or* $\gamma = 0$.

First case: $\gamma = 0$.

By the assertion (3) of the under-claim 1, we have:

$$\begin{cases} a^{x\delta}\alpha^{\delta(2-\delta)}(\ln(a))^{\delta(2-\delta)} + b^{y\delta}\beta^{\delta(2-\delta)}(\ln(b))^{\delta(2-\delta)} = c^{z\delta}\gamma^{\delta(2-\delta)}(\ln(c))^{\delta(2-\delta)} \\ \gamma = 0 \\ a^{x\delta}(\ln(a))^{\delta(2-\delta)} > 0 \text{ and } b^{y\delta}(\ln(b))^{\delta(2-\delta)} > 0 \\ \alpha \geq 0 \text{ and } \beta \geq 0 \end{cases} \Rightarrow \alpha = \beta = \gamma = 0.$$

Second case: $\alpha = 0$.

*We have: $b^{y\delta}\beta^{\delta(2-\delta)}(\ln(b))^{\delta(2-\delta)} = c^{z\delta}\gamma^{\delta(2-\delta)}(\ln(c))^{\delta(2-\delta)}$.

*i.e.:

$b^y\beta^{2-\delta}(\ln(\mathrm{b}))^{2-\delta} = c^z\gamma^{2-\delta}(\ln(\mathrm{c}))^{2-\delta}$ or $b^{\frac{y}{2-\delta}}\beta\ln(b) = c^{\frac{z}{2-\delta}}\beta\ln(c)$ or $b^{\beta b^{\frac{y}{2-\delta}}} = c^{\gamma c^{\frac{z}{2-\delta}}}$ or $b^\beta = c^{\gamma\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}}$.

*But by the Bezout theorem: $\gcd(a,b,c) = 1 \Rightarrow \exists u, v$ two integers such that:

$$\begin{cases} v > u > 0 \\ -uc^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)} + vb^\beta = 1 \end{cases}$$

*So: $-uc^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)} + vb^\beta = 1 = -uc^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)} + vc^{\gamma\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}} = c^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)}\left(vc^{\gamma\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}} - E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)\right)} - u\right)$.

*But: $\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}} - E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)\right) \geq 0$ *and* $v > 0 \Rightarrow 1 \geq c^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)}(v-u) > 0$.

*That is, working with integers, $1 = c^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)}(v-u)$ i.e. $1 = c^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)} = v - u$.

*So: $\begin{cases} 1 = c^{\gamma E\left(\left(\frac{c^z}{b^y}\right)^{\frac{1}{2-\delta}}\right)} \\ c > 2 \end{cases} \Rightarrow \gamma = 0 \Rightarrow \beta = \gamma = \alpha = 0.$

Third case: $\beta = 0$.

By analogy the third case is obtained exactly as the second case. So: $\alpha = \beta = \gamma = 0$.

Return to the proof of claim3:

 ISSN 2349-0292
Impact Factor 3.802

*We have: $\begin{cases} a^{m+\alpha} + b^{\beta+m} = c^{\gamma+m} \\ \alpha = \beta = \gamma = 0 \\ \text{m} = 3{,}4 \text{ or } 5 \end{cases} \Rightarrow \begin{cases} a^m + b^m = c^m \\ m = 3{,}4 \text{ } or \text{ } 5 \end{cases}$.

*But, by the Euler-Gauss theorem, this is impossible.

*So, necessarily, we have: $m = \min(x, y, z) = 2$.

*This ends the deduction of the Beal conjecture from the abc-conjecture.

Second method: See S. Laishram and T.N. Shorey [56] for a second method.

3. THE FERMAT CATALAN CONJECTURE:

Corollary 3: (see [17], [111], [112]) Let A, B, C be integers then the abc-conjecture implies that the below set is finite:

$$T_{A,B,C} = \{(a,b,c,x,y,z) \in \mathbb{N}^6, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1, abc \neq 0, a^x < \mathrm{b}^y < \mathrm{c}^z, \gcd(Aa^x, Bb^y, Cc^z) = 1, Aa^x + Bb^y = Cc^z\}$$

Proof: (of corollary 3)

Claim 5: Let p, q, r integers≥ 2, we have:$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \Rightarrow \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{41}{42}$.

Proof: (of calim5)

*We can suppose: $2 \leq p \leq q \leq r$.

***First case:** $p = 2$.

We have: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \Rightarrow \frac{1}{q} + \frac{1}{r} < 1 - \frac{1}{p} = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow r \geq q \geq 3$.

Remark: if: $2 \leq q < 3$, we have: $q = 2$, so: $\frac{1}{q} + \frac{1}{r} = \frac{1}{2} + \frac{1}{r} < \frac{1}{2} \Rightarrow \frac{1}{r} < 0$ which is impossible.

First under-case: $q = 3$.

*$\frac{1}{q} + \frac{1}{r} = \frac{1}{3} + \frac{1}{r} < \frac{1}{2} \Rightarrow \frac{1}{r} < \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \Rightarrow r > 6 \Rightarrow r \geq 7$.

*So: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$.

Second under-case: $q = 4$.

*$\frac{1}{q} + \frac{1}{r} = \frac{1}{4} + \frac{1}{r} < \frac{1}{2} \Rightarrow \frac{1}{r} < \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow r > 4 \Rightarrow r \geq 5$.

*So: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < \frac{41}{42}$.

Third under-case: $q \geq 5$.

*We have: $5 \leq q \leq r$.

*So:$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2} + \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} + \frac{2}{5} = \frac{9}{10} < \frac{41}{42}$.

***Second case:** $p = 3$.

First under-case: $q = 3$.

*We have: $\frac{1}{r} < 1 - \frac{1}{p} - \frac{1}{q} = 1 - \frac{2}{3} = \frac{1}{3} \Rightarrow r > 3 \Rightarrow r \geq 4$.

*So: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{3} + \frac{1}{3} + \frac{1}{r} \leq \frac{2}{3} + \frac{1}{4} = \frac{11}{12} < \frac{41}{42}$.

Second under-case: $q \geq 4$.

*We have: $4 \leq q \leq r$.

*So: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{3} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{3} + \frac{2}{4} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < \frac{41}{42}$.

***Third case:** $p \geq 4$.

*We have: $r \geq q \geq p \geq 4$.

*So: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{3}{4} < \frac{41}{42}$.

Claim6: $\exists L > 0 \forall (a, b, c, x, y, z) \in T_{A,B,C}$ we have: $a^x < b^y < c^z \leq L$.

Proof: (of claim6)

*Let: $\frac{1}{41} > \epsilon > 0\ \exists K > 0$ such that for the abc-triple(Aa^x, Bb^y, Cc^z), we have:

$\min(|A|, |B|, |C|)\max(a^x, b^y, c^z) = \min(|A|, |B|, |C|)c^z \leq \max(|A|a^x, |B|b^y, |C|c^z) < K\left(r(ABCa^x b^y c^z)\right)^{1+\epsilon}$
$< K|ABC|^{1+\epsilon}(abc)^{1+\epsilon}$

$< K|ABC|^{1+\epsilon}(c^{\frac{z}{x}} c^{\frac{z}{y}} c)^{1+\epsilon} = K|ABC|^{1+\epsilon} c^{(1+\epsilon)\left(1+z\left(\frac{1}{x}+\frac{1}{y}\right)\right)}$.

*So: $c^{z - z(1+\epsilon)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} < \frac{|ABC|^{1+\epsilon}}{\min(|A|,|B|,|C|)} K$.

*But: by claim5, we have:

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{41}{42} \Rightarrow z - z(1+\epsilon)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq z - \frac{41}{42} z(1+\epsilon) = \frac{1}{42} - \frac{41\epsilon}{42}$

$\Rightarrow c^{z\left(\frac{1}{42} - \frac{41\epsilon}{42}\right)} < \frac{|ABC|^{1+\epsilon}}{\min(|A|,|B|,|C|)} K$.

*So: $\frac{1}{42} - \frac{41\epsilon}{42} > 0 \Rightarrow a^x < b^y < c^z < \left(\frac{|ABC|^{1+\epsilon}}{\min(|A|,|B|,|C|)} K\right)^{\frac{1}{\frac{1}{42} - \frac{41\epsilon}{42}}} = L$.

Conclusion: So, L being an absolute constant, there are a finite number of abc-triples (a, b, c) and a finite number of triples (x, y, z), such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, resolving the Diophantine equation $Aa^x + Bb^y = Cc^z$.

Corollary4 :(The Fermat–Catalan conjecture) The Below set is finite:

$$T = \{(a, b, c, x, y, z) \in \mathbb{N}^6, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1, abc \neq 0, \gcd(a, b, c) = 1, a^x + b^y = c^z\}$$

Proof: (of Corollary 4)

The result follows from corollary 3 for $A = B = C = 1$.

Remark: When we don't allow (x, y, z) varying, Henri Darmon and Andrew Granville showed (see [21]) in 1994, by a proof none based on the abc-conjecture, the below theorem:

Theorem2: (Darmon-Granville [21]) let x, y, z positive integers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<1$ and A, B, C fixed integers. Then: the set $T_{x,y,z,A,B,C} = \{(a, b, c) \in \mathbb{N}^3, abc \neq 0, \gcd(Aa, Bb) = 1, Aa^x + Bb^y = Cc^z\}$ is finite.

4. THE ROTH THEOREM:

Corollary 5: (Roth theorem on Diophantine approximation of algebraic numbers [88]) Let α an algebraic number over $\mathbb{Q}$ and $\epsilon > 0$, then the abc-conjecture implies that: $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$ for only finitely many rational numbers.

Remark: (1) Klaus Roth (1925-2015) is a German-born British mathematician.

(2) This theorem was proved by Roth in 1955 (see [88]). This proof earned Roth the Fields prize.

(3)See also Machiel Van Frankenhuysen [37].

Proof: (of corollary 5)

*Let α an algebraic number over $\mathbb{Q}$ and $\epsilon > 0$.

*Show that the set R= $\{(p, q) \in \mathbb{Z}^* \times \mathbb{N}^*$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$ $\}$ is finite.

*Suppose contrarily that R contains an infinite sequence (p_n, q_n) with $p_n \in \mathbb{Z}^*$ and $q_n \in \mathbb{N}^*$.

*Denote by $E(x)$ the integer part of the real x i.e. the single integer $E(x)$ such that: $E(x) \leq x < E(x) + 1$.

*$(p_n, q_n) \in R \Rightarrow \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{2+\epsilon}} \Rightarrow q_n \leq q_n^{\frac{4}{7}(2+\epsilon)} < (\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|})^{\frac{4}{7}} \Rightarrow q_n \leq E\left((\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|})^{\frac{4}{7}}\right)$.

*Consider the abc-triples $(1, E(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}) - 1, E(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}})$ i.e. such that:

$$\begin{cases} \gcd\left(1, E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right) - 1, E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)\right) = 1 \\ 1 + (E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right) - 1) = E(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}) \end{cases}.$$

*By the Backer abc-conjecture version (see the assertion (i) of proposition 9), applied to the abc-triple $(1, E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right) - 1, E(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}))$ we have:

$$q_n \leq E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right) < (r\left(\left(E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right) - 1\right) E\left(\left(\frac{1}{\left|\alpha - \frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)\right))^{\frac{7}{4}}.$$

[Ghanim *et al.*, 9(2): February, 2022] ISSN 2349-0292
Impact Factor 3.802

*By the quality version of the abc-conjecture (see the assertion (iii) of proposition 9) applied to the abc-triple $\left(1, E\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)-1, E\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)\right)$, there are a finite number of such triples satisfying:

$$\frac{\ln\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)}{\ln(q_n)}=\frac{\ln\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)}{\frac{4}{7}\ln(q_n)}\geq\frac{\ln\left(E\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)\right)}{\frac{4}{7}\ln\left(\left(r\left(\left(E\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)-1\right)E\left(\left(\frac{1}{\left|\alpha-\frac{p_n}{q_n}\right|}\right)^{\frac{4}{7}}\right)\right)\right)^{\frac{7}{4}}\right)}>1+\epsilon.$$

*That is: there is a finite number of rational numbers: $\frac{p_n}{q_n}, q_n>0$ such that: $\frac{1}{{q_n}^{2+\epsilon}}>\left|\alpha-\frac{p_n}{q_n}\right|$.

*Finally, this contradicting our starting absurd hypothesis (assuring that the sequence (p_n, q_n) is infinite), the proof is finished.

5. THE WIEFERICH-SILVERMAN THEOREM:

Definition7: (Wieferich number [114]) For: $a \in \mathbb{N}, a \geq 2$: a Wieferich number in the base a is a prime p such that: p^2 divides $a^{p-1}-1$.

Example: The sole Wieferich primes $< 4.10^{12}$ are: 1093 and 3511.

Remark: (1) Arthur Josef Alwin Wieferich (1884-1954) is a German Mathematician having some works in Number theory.

(2) Wieferich defined his numbers during 1909 in his works on the Fermat last theorem.

Corollary6: (Wieferich [114]-Silverman [98] theorem (see also [17], [93], [111], [112]))$\forall a \in \mathbb{N}^*-\{1\}$, the abc-conjecture implies that the set: $U(a)=\{p \in \mathbb{P}, p^2$ doas not divide $a^{p-1}-1\}$ is infinite.

Remark: Silverman showed this theorem in 1988 [98].

Proof: (of corollary 6)

*Let a an integer≥ 2 and suppose contrarily that $U(a)$ is finite.

*Let n an arbitrarily integer.

Claim5: $\exists(\alpha(p))_{p\in\mathbb{P}} \geq 0 \exists u_n=\prod_{p\in U(a)} p^{\alpha(p)}$ and $v_n=\prod_{p\in\mathbb{P}-U(a)} p^{\alpha(p)}$ such that $a^n-1=u_n v_n$.

Where: the set $\{p \in \mathbb{P}-U(a), \alpha(p) \neq 0\}$ is finite.

Proof: (of claim5)

The result follows by the arithmetical fundamental theorem.

Claim 6: $\exists L>0$ an absolute constant such that $r(u_n)=$ radical of $u_n \leq L$.

Proof: (of claim7)

$*$ $U(a)$ being finite, it is sufficient to take: $L=\prod_{p\in U(a)} p$.

*We have well: $r(u_n) \leq L$.

Claim7: We have: a prime integer p divides $v_n \Rightarrow p^2$ divides v_n.

Proof: (of claim7)

Remark: given an integer d and a positive integer m such that $\gcd(d, m) = 1$, the multiplicative order of d modulo m is $o_m(d)$ =min $(\{k \in \mathbb{N}^*, d^k \equiv 1\ (modulo\ m)\}$.

*By claim 5: $\begin{cases} a^n - 1 = u_n v_n \\ p \text{ prime dividing } v_n \end{cases} \Rightarrow p$ das not divide $a \Rightarrow \gcd(p, a) = 1$.

*Let: $m_1 = o_p(a)$ and $m_2 = o_{p^2}(a)$.

*We have :

$$\begin{cases} a^{m_1} = 1 + \delta p \\ p \text{ prime divides } C_p^k = \frac{p!}{k!(p-k)!}\ for\ 1 \leq k \leq p-1 \end{cases} \Rightarrow a^{m_1 p} = (1 + \delta p)^p = 1 + \sum_{k=1}^{p} C_p^k (\delta p)^k \equiv 1(modulo\ p^2).$$

*So: $m_2 = o_{p^2}(a) \Rightarrow m_2$ divides $m_1 p$.

*But: $m_2 = o_{p^2}(a) \Rightarrow a^{m_2} \equiv 1\ (modulo\ p^2) \Rightarrow a^{m_2} \equiv 1(modulo\ p)$.

*So: $m_1 = o_p(a) \Rightarrow m_1$ divides m_2.

*We have:

$$\begin{cases} m_1 p = s m_2 \\ m_2 = t m_1 \end{cases} \Rightarrow m_2 p = t(m_1 p) = s t m_2 \Rightarrow p = st, p \text{ being prime} \Rightarrow (s = 1\ and\ t = p) or (s = p\ and\ t = 1)$$

$\Rightarrow m_1 = m_2$ Or $m_2 = p m_1$.

First case: $m_2 = p m_1$.

*By claim5: p divides $v_n \Rightarrow p \in \mathbb{P} - U(a) \Rightarrow a^{p-1} \equiv 1(modulo\ p^2)$.

*So: $m_2 = o_{p^2}(a) \Rightarrow m_2$ divides $p - 1$.

*So: $p - 1 = l m_2 = l p m_1 \Rightarrow p\ divdes\ p - 1 \Rightarrow p = 1$.

*This contradicting the fact that p is a prime integer≥ 2, this case cannot occur.

Second case: $m_2 = m_1$.

*By claim6, we have: p divides $v_n \Rightarrow p\ divides\ a^n - 1 \Rightarrow a^n \equiv 1\ (modulo\ p)$.

*So: $m_1 = o_p(a) \Rightarrow m_1$ divides n.

*So: $m_2 = m_1$ divides $n \Rightarrow n = r m_2$.

*So, we have:

$$a^{m_2} = 1 + \sigma p^2 \Rightarrow a^n = a^{r m_2} = (1 + \sigma p^2)^r = 1 + \sum_{k=1}^{r} C_r^k (\sigma p^2)^k = 1 + p^2 \sum_{k=1}^{r} C_r^k \sigma^k p^{2k-2} \equiv 1\ (modulo\ p^2).$$

*Finally: $p\ divides\ v_n$ and $p^2\ divides\ a^n - 1 = u_n v_n \Rightarrow p^2$ divides v_n.

 ISSN 2349-0292
Impact Factor 3.802

Claim8: We have: $r(v_n) \leq v_n^{\frac{1}{2}}$.

Proof: (of claim8)

By claim 7: $v_n = \prod_{p\in\mathbb{P}-U(a)} p^{\alpha(p)}$ with $\alpha(p) = 0$ or $\alpha(p) \geq 2 \Rightarrow v_n^{\frac{1}{2}} = \prod_{p\in\mathbb{P}-U(a)} p^{\frac{1}{2}\alpha(p)} \geq r(v_n)$.

Claim9: (the Wanted contradiction) $\exists K$ an absolute constant such that $v_n \leq K$.

Proof: (of claim9)

*Applying the abc-Conjecture to the abc-triple $(1, u_n v_n, a^n)$, for $0 < \epsilon < 1$, we have: $\exists K(\epsilon) > 0$ such that:

$v_n \leq \max(1, u_n v_n, a^n) \leq K(\epsilon) r(u_n v_n a^n)^{1+\epsilon} = K(\epsilon)(r(u_n))^{1+\epsilon}(r(v_n))^{1+\epsilon}(r(a^n))^{1+\epsilon} =$
$K(\epsilon)\left(r(u_n)\right)^{1+\epsilon}\left(r(v_n)\right)^{1+\epsilon}\left(r(a)\right)^{1+\epsilon}$.

*So, by claim6, claim8:

$v_n \leq K(\epsilon)L^{1+\epsilon} v_n^{\frac{1+\epsilon}{2}}(r(a))^{1+\epsilon}$ i.e. $v_n^{\frac{1-\epsilon}{2}} \leq K(\epsilon)L^{1+\epsilon}(r(a))^{1+\epsilon}$ i.e.$v_n \leq K = (K(\epsilon)L^{1+\epsilon}(r(a))^{1+\epsilon})^{\frac{2}{1-\epsilon}}$

Conclusion:*Tending $n \to +\infty \Rightarrow a^n \to +\infty \Rightarrow v_n = \frac{a^n - 1}{u_n} \to +\infty$ (Because u_n is bounded).

* So, K being an absolute constant, the contradiction, showing that $U(a)$ is infinite, is reached.

6. THE ERDOS-WOODS CONJECTURE:

Corollary7: (Erdos [30]-Woods [133] conjecture (see also [17], [93], [111], [112]) the abc-conjecture implies that: $\exists k \in \mathbb{N}^*$ such that:

$$\forall x, y \in \mathbb{N}^* \forall i \in \{0,1, \ldots, k-1\} r(x+i) = r(y+i) \Rightarrow x = y$$

Remark: 1) Paul Erdos (1913-1996) is a Hungarian mathematician.

Proof: (of corollary 7)

*Suppose contrarily that: $\forall k \in \mathbb{N}^* \exists x_k < y_k$ such that $\forall i \in \{0,1, \ldots, k-1\} r(x+i) = r(y+i)$.

*In particular for $k \geq 4$, we have: $\{0,1,2,3\} \subset \{0, 1, 2, \ldots, k-1\}$.

*We have:$\forall i \in \{0,1,2,3, \ldots, k-1\}$ $r(x_k + i) = r(y_k + i) \Rightarrow \begin{cases} x_k + i = \prod_{j=1}^{n_i} p_{ij}^{\alpha_{ij}} \\ y_k + i = \prod_{j=1}^{n_i} p_{ij}^{\beta_{ij}} \end{cases}$.

Claim10: we have: $\forall k \geq 4$ we have: $x_k \geq k$.

Proof: (of claim10)

*Suppose contrarily that $\exists k \geq 4$ $such\ that$ $x_k < k$ i.e. $x_k \in \{0, 1, 2, \ldots, k-1\}$.

*So, by the absurd hypothesis, applied for $i = x_k$ we have: $r(x_k + x_k) = r(2x_k) = r(x_k + y_k)$.

*So:$\begin{cases} 2x_k = \prod_{j=1}^{n_k} p_{kj}^{\alpha_{kj}} \\ y_k + x_k = \prod_{j=1}^{n_k} p_{kj}^{\beta_{kj}} \Rightarrow \prod_{j=1}^{n_k} p_{kj}^{\alpha_{kj}} + \prod_{j=1}^{n_k} p_{kj}^{\gamma_{kj}} = 2\prod_{j=1}^{n_k} p_{kj}^{\beta_{kj}}. \\ 2y_k = \prod_{j=1}^{n_k} p_{kj}^{\gamma_{kj}} \end{cases}$

$\Rightarrow \prod_{j=1}^{n_k} {p_{kj}}^{\alpha_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} + \prod_{j=1}^{n_k} {p_{kj}}^{\gamma_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} = 2\prod_{j=1}^{n_k} {p_{kj}}^{\beta_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})}$.

*Suppose that: $\exists j \in \{1,2,\dots,n_k\}$ such that: $\beta_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) \neq 0$.

*So: $\gamma_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = 0 \;\; or \;\; \alpha_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = 0$.

*This being impossible in all the cases, we have: $\forall j \in \{1,2,\dots,n_k\}\;\; \beta_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = 0$.

*So: $\prod_{j=1}^{n_k} {p_{kj}}^{\alpha_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} + \prod_{j=1}^{n_k} {p_{kj}}^{\gamma_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} = 2$.

*So: $\prod_{j=1}^{n_k} {p_{kj}}^{\alpha_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} = \prod_{j=1}^{n_k} {p_{kj}}^{\gamma_{kj}-\min(\alpha_{kj},\beta_{kj},\gamma_{kj})} = 1$.

*So: $\forall j \in \{1,2,\dots,n_k\}\;\; \alpha_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = \beta_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = \gamma_{kj} - \min(\alpha_{kj},\beta_{kj},\gamma_{kj}) = 0$.

*That is: $\forall j \in \{1,2,\dots,n_k\}\;\; \alpha_{kj} = \gamma_{kj} = \beta_{kj}$.

*So: $x_k = y_k$.

*This being impossible, claim10 is showed.

<u>Claim11:</u> $r(st)r(\gcd(s,t)) = r(s)r(t)$.

<u>Proof:</u> (of claim11)

Because $r(\gcd(s,t))$ appears ones in $r(st)$ and appears twice in $r(s)r(t)$.

<u>Claim12:</u> We have: $r(\gcd(y_k(y_k+1),(y_k+2)(y_k+3))) \leq 6$.

<u>Proof:</u> (of claim12)

*$\gcd(y_k(y_k+1),(y_k+2)(y_k+3)) = \prod_{p\in\mathbb{P}} p^{\alpha(p)}$: the set $\{p \in \mathbb{P}, \alpha(p) \neq 0\}$ being finite.

*Let p is a prime integer, we have:

p divides $y_k(y_k+1)$ and p divides$(y_k+2)(y_k+3) \Leftrightarrow (p$ divides $\mathrm{y_k}$ or p divides $\mathrm{y_k}+1)$and$(p$ divides $\mathrm{y_k}+2$ or p divides $\mathrm{y_k}+3)$.

$\Leftrightarrow (p$ divides $\mathrm{y_k}$ and p divides $\mathrm{y_k}+2)$or $(p$ divides $\mathrm{y_k}$ and p divides $\mathrm{y_k}+3)$or $(p$ divides $\mathrm{y_k}+1$ and p divides $\mathrm{y_k}+2)$or$(p$ divides $\mathrm{y_k}+1$ and p divides $\mathrm{y_k}+3)$.

$\Leftrightarrow p$ divides 2 or p divides $3 \Leftrightarrow p = 2 \; or \; p = 3 \Leftrightarrow \gcd(y_k(y_k+1),(y_k+2)(y_k+3)) = 2^\alpha 3^\beta$.

*So: $r(\gcd(y_k(y_k+1),(y_k+2)(y_k+3))) \leq 6$.

<u>Claim13:</u> We have:$r(y_k(y_k+1)r((y_k+2)(y_k+3)) \leq 6(y_k - x_k)$.

<u>Proof:</u> (of claim13)

*We have: $\forall i \in \{0,1,2,\dots,k-1\}\;\; \forall j \in \{1,2,\dots,n_i\}\;\; p_{ij}$ divides: $y_k - x_k = (y_k+i) - (x_k+i)$.

*By claim11 and claim 12, we have:

*So: $r(y_k(y_k+1))r((y_k+2)(y_k+3)) = r(\gcd((y_k(y_k+1),(y_k+2)(y_k+1))r((y_k(y_k+1)(y_k+2)(y_k+1)) \leq 6(y_k - x_k)$.

Claim14: $\forall 1 > \epsilon > 0 \exists K(\epsilon)$such that $y_k < (6^{1+\epsilon}(K(\epsilon))^2)^{\frac{1}{1-\epsilon}}$.

Proof: (of claim14)

*Recall that, In particular for $k \geq 4$, we have: $\{0,1,2,3\} \subset \{0, 1, 2, \ldots, k-1\}$.

*$\forall 1 > \epsilon > 0 \exists K(\epsilon)$ Such that, applying the abc-conjecture to the abc-triples: $\begin{cases} (a,b,c) = (1, y_k, y_k+1) \\ (a,b,c) = (1, y_k+2, y_k+3) \end{cases}$, we have:

$$\begin{cases} y_k + 1 \leq K(\epsilon)(r(y_k(y_k+1))^{1+\epsilon} \\ y_k + 3 \leq K(\epsilon)(r((y_k+2)(y_k+3)))^{1+\epsilon} \end{cases}.$$

*So: $y_k{}^2 \leq (y_k+1)(y_k+3) \leq (K(\epsilon))^2 (r(y_k(y_k+1))^{1+\epsilon} (r((y_k+2)(y_k+3)))^{1+\epsilon}$.

*So, by corollary 12, we have:

$$y_k{}^2 \leq (K(\epsilon))^2 (6(y_k - x_k))^{1+\epsilon} < (K(\epsilon))^2 (6y_k)^{1+\epsilon}.$$

*That is: $y_k < (6^{1+\epsilon}(K(\epsilon))^2)^{\frac{1}{1-\epsilon}}$.

Conclusion: claim 10 and claim 14 give the wanted contradiction, the relation $k \leq x_k < y_k < (6^{1+\epsilon}(K(\epsilon))^2)^{\frac{1}{1-\epsilon}}$ meaning that the sequence (y_k) is both bounded and unbounded.

Corollary8 :(The Erdős–Woods conjecture[30]) $\exists k$ an integer>1 such that $\forall a, b$ integers such that $a > 1$ and $b > a + k$: lcm(a,a+1,…, a+k) and lcm(b,b+1,…., b+k) have not commune prime factors, where lcm denotes the lower commune multiple.

7. ERDOS CONJECTURE ON $2^n - 1$:

Corollary 9: (Erdos conjecture [29]) the abc-conjecture implies that the greatest prime factor $P(2^n - 1)$ of $2^n - 1$ is such that: $\lim_{n \to +\infty} \frac{P(2^n-1)}{n} = +\infty$

Remark: this is the Erdos conjecture conjectured in 1965.

Remark: According to Waldschmidt [111], [112]:"in 2002, R. Murty and S. Wong [76] showed that the Erdos conjecture is a consequence of the abc-conjecture. In 2012, C.L. Stewart [104] showed the Erdos conjecture (in the more general frame of the Lucas and Lehmer sequences) by proving that:$\frac{P(2^n-1)}{n} > e^{\frac{\ln(n)}{104 \ln(\ln(n))}}$"

Proof: (of corollary 9)

See, for a proof, Murty R. -Wong S. (2002) [75] and L.C. Stewart (2012) [103]

8. THE BROCARD PROBLEM:

The Henri Brocard problem: is the resolution, in (n, m), of the Diophantine equation: $n! + 1 = m^2$

Remark: This problem was formulated independently by the French Mathematician Pierre-René-Jean-Baptiste Henri Brocard (1845-1922) in 1876 and 1885 (in his two papers [12], [13]) and in 1913 by the Indian Mathematician Srinivasa Ramanujan (1887-1920) (in his papers [86], [87])

Definition8: the solutions $(n, m) \in \mathbb{N}^2$ of the Diophantine equation: $n! + 1 = m^2$ are called Brown numbers.

ISSN 2349-0292
Impact Factor 3.802

Proposition15: The sole known Brown numbers are the pairs: (4, 5); (5, 11) and (7, 71)

Conjecture: (of Paul Erdos [30]-S. Ramanujan [86], [87]) there no other Brown numbers other than (4, 5); (5, 11) and (7, 71). Ramanujan Formulated the problem in the form "the number 1+n! is a perfect square for the values 4, 5, 7 of n. Find other values"

Corollary 9: (Overholt (1993) [82] the abc-conjecture implies that the set $B = \{(n, m) \in \mathbb{N}^2, n! + 1 = m^2\}$is finite.

Proposition 16: there are no supplementary solutions to the Brocard problem for $n \leq 10^9$ (Berndt-Galway [7]), for $n \leq 10^{12}$(Matson [67] in 1917) and for $n \leq 10^{15}$(Epstein-Glickman [27] in 2020)

Corollary10 :(Dąbrowski conjecture (1996) [19]) the abc-conjecture implies that: $\forall A \in \mathbb{N}$ the set $\mathrm{B}_A = \{(\mathrm{n}, \mathrm{m}) \in \mathbb{N} \times \mathbb{N},\ n! + A = m^2\}$ is finite.

Remark: the solution giving the largest n is 11! $+18^2 = 6318^2$.

Corollary11: (Cushinge-Pascoe (2016) [18]) the abc-conjecture implies that: $\forall A \in \mathbb{N}$ the set $B_A = \{(n, m) \in \mathbb{N}^2, \exists (a, b) \in \mathbb{N}^2, \text{such that}: \mathrm{m} = a^2 b^3 \text{ and } n! + A = m\}$ is finite.

Corollary12: (Luca conjecture (2002) [62]) the abc-conjecture implies that for any polynomial P having integer coefficients and a degree≥ 2, the set: $B_P = \{(n, m) \in \mathbb{N}^2, n! = P(m)\}$ is finite.

Proof: (of corollary 12)

Remark: (i) Berend and Osgood ([6]) showed that the density of the set of positive integers n for which there exists an integer m such that $n! = P(m)$ is zero.

(ii) If P(X) =X^d, the equation $n! = P(m)$ has no solutions with |m|>1.

(ii) If $P(X) = X^d \pm 1$ *and* $d \geq 3$, Erdos and Oblath ([28]) showed that the equation:$n! = P(m)$ has no solutions with |m|>1.

Let $P(X) = \sum_{k=0}^{d} a_{d-k} X^k$ with: $a_i \in \mathbb{N}$ *and* $d \geq 2$ and consider the Diophantine equation $n! = P(m)$ in (n, m).

Claim 15: We have: $\exists (c, b_i, k) \in \mathbb{N}^3$ such that: $k^d + b_1 k^{d-1} + \dots + b_d = cn!$.

Proof: (of claim15)

*The result is obtained by multiplication of the Diophantine equation by: $d^d {a_0}^{d-1}$.

*So:$c = d^d {a_0}^{d-1}$, $k = a_0 dm$ and $b_i = d^i {a_0}^{i-1}$ for: $i = 1,2, \dots, d$.

Claim16: We have: $\exists (z, c_i) \in \mathbb{N}^2$ such that: $Q(z) = z^d + c_2 z^{d-2} + \dots + c_d = cn!$.

Proof: (of claim16)

*b_1 being a multiple of d we have: $z = k + \frac{b_1}{d} \in \mathbb{N}$.

*c_i are integer coefficients easily computed in terms of a_i and d for $i = 2, 3, \dots, n$.

Claim17: When $|z|$ is large we have: $\frac{|z|}{2} < |Q(z)| < 2|z|$.

Claim18: We have:$\exists L_1, L_2 > 0$ two constants depending only on the coefficients a_i for $i = 0,1,2,\dots,d$ of P such that: $|d\ln(|z|) - \ln(n!)| < L_1$ for: $|z| > L_2$, when (n,z) are solutions of the equation of claim 16.

Proof: (of claim18)

The result follows by combination of claim16 and claim17.

Claim19: (i) We can suppose $L_2 > L_1$ (ii) So if (n,z) are integers satisfying the inequalities of claim18, then: $n > c$.

Let $Q(X) = X^d + R(X)$.

First Case: if $R(X) = 0$.

The equation of claim 16 reduces to $z^d = cn!$.

Claim20: The equation $z^d = cn!$ has no integer solutions (n,z) for $n > 2c$.

Proof: (of claim20)

*When: $n > 2c$: the interval $]\frac{n}{2}, n[$ contains a prime larger than c which will appear at the exponent 1 in the product $cn!$.

*So: $cn!$ cannot be a perfect power.

*So, by the inequalities of claim 18, the equation: $n! = P(m)$ has only finitely many solutions in this first case.

Second case: if $R(X) \neq 0$.

Let $j \leq d$ the largest integer with $c_j \neq 0$, we have:

Claim21: $z^j + c_2 z^{j-2} + + \cdots + c_j = \frac{cn!}{z^{d-j}}$.

Let: $R_1(X) = \frac{R(X)}{X^{d-j}} = c_2 z^{j-2} + + \cdots + c_j \in \mathbb{Z}[X]$, we have: $z^j + R_1(z) = \frac{cn!}{z^{d-j}}$.

Claim22: $\exists L_3, L_4 \geq L_2$ such that: $0 < |R_1(z)| < L_3 |z|^{j-2}$ when $|z| > L_4$.

Proof: (of claim22)

We can take: $L_3 = 1 + |c_2|$.

Claim23: (i) if $D = \gcd(z^d, R_1(z))$, then all the prime divisors of D divide c_j.

(ii) So, we have: $\frac{z^j}{D} + \frac{R_1(z)}{D} = \frac{cn!}{z^{d-j}D}$.

(iii) $(A, B, C) = (\frac{z^j}{D}, \frac{R_1(z)}{D}, \frac{cn!}{z^{d-j}D})$ is an abc-triple.

Proof: (of claim23)

The result follows easily by definition of $R_1(z)$.

Claim24: (Alborghetti [2]) We have: $\prod_{p \le n} p \le 4^n$, the product considered for all prime integers $p \le n$ (the relation is true for all $n \ge 1$.

Proof: (of claim24)

For a proof see Alborghetti 2011 [2] pp 15-18.

Claim25: We have:

(i) $\forall \epsilon > 0 \exists L_5 > 0$ (a constant depending of ϵ only) such that: $\frac{|z|^j}{D} < L_5 (r\left(\frac{z^j R_1(z) cn!}{D^3}\right))^{1+\epsilon}$.

(ii) $r\left(\frac{z^j}{D}\right) \le r(z^j) \le |z|$.

(iii) $r\left(\frac{R_1(z)}{D}\right) \le \frac{|R_1(z)|}{D} \le \frac{L_3 |z|^{j-2}}{D}$.

(iv) $r\left(\frac{cn!}{D}\right) \le r(cn!) = r(n!) = \prod_{p \le n} p \le 4^n$ (because : $c > n$).

(v) $r\left(\frac{z^j R_1(z) cn!}{D^3}\right) \le r\left(\frac{z^j}{D}\right) r\left(\frac{R_1(z)}{D}\right) r\left(\frac{cn!}{D}\right) \le \frac{L_3 |z|^{j-1} 4^n}{D}$.

(vi) For $L_6 = L_5 L_3^{1+\epsilon}$, we have: $\frac{|z|^j}{D} \le L_6 (\frac{|z|^{j-1} 4^n}{D})^{1+\epsilon}$.

(vii) $|z|^j \le L_6 \frac{|z|^{(j-1)(1+\epsilon)} 4^{(1+\epsilon)n}}{D^\epsilon}$ or $|z|^{1+\epsilon-\epsilon j} \le L_6 4^{n(1+\epsilon)}$.

(viii) Taking $\epsilon = \frac{1}{2d} \le \frac{1}{2j}$, $L_7 = 2(1+\epsilon)\ln(4)$, $L_8 = 2\ln(L_6)$, $L_9 = dL_7$ and $L_{10} = dL_8$ we have: $|z|^{\frac{1}{2}} \le |z|^{1+\epsilon-\epsilon j} \le L_6 4^{n(1+\epsilon)}$ or $\ln(|z|) < L_7 n + L_8$ or $d\ln(|z|) \le nL_9 + L_{10}$.

(ix) If $L_{11} = L_1 + L_{10}$, we have: $|d\ln(|z|) - \ln(n!)| < L_1$ for $|z| > L_2 \Rightarrow \ln(n!) \le L_1 + d\ln(|z|) \le nL_9 + L_{11}$.

Proof: (of claim25)

(i)The result follows by application of the abc-conjecture to the abc-triple $(A, B, C) = \left(\frac{z^j}{D}, \frac{R_1(z)}{D}, \frac{cn!}{z^{d-j}D}\right)$.

The assertions (ii)-(ix) follow successively.

 ISSN 2349-0292
Impact Factor 3.802

Claim26: (Stirling formula for approximating $n!$)[121]$\forall n \in \mathbb{N}^* \ \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$.

Claim27: $n < L_{12}$ and $|z| < L_{13}$.

Proof: (of claim 27)

*The result follows by claim26 and the inequality: $|d\ln(|z|) - \ln(n!)| < L_1$ for $|z| > L_2$.

*$nL_9 + L_{11} \geq \ln(n!) \geq \ln(\sqrt{2\pi}) + \left(n + \frac{1}{2}\right)\ln(n) - n$.

*That is: $L_{11} - \ln(\sqrt{2\pi}) \geq n(\ln(n) - 1 - L_9)$.

*So, for: $n \geq e^{2+L_9}$, we have: $n \leq L_{12} = L_{11} - \ln(\sqrt{2\pi})$ and $d\ln(|z|) \leq nL_9 + L_{10} \leq L_9L_{12} + L_{10}$ or $|z| \leq e^{\frac{L_9L_{12}+L_{10}}{d}} = L_{13}$.

Conclusion: (i) The equation $n! = P(m)$ has only finitely many integer solutions (n, m).

(ii)Corollaries 9, 10 and 11 follow from corollary 12.

Remark: For more details on Brocard Problem See Gérardin 1906 [39].

9. THE MORDELL CONJECTURE:

Definition9: a plane projective algebraic curve of degree n is a set of points $M(x, y)$ whose coordinates satisfy an algebraic equation $P_n(x, y) = 0$ where $P_n(x, y)$ is a polynomial of degree n.

Definition10: the order of a plane projective algebraic curve E is the number of intersection points of this curve with any line lying in the plane of this curve.

Definition11: If E is a plane projective algebraic curve defined by an irreducible homogenous polynomial of degree m, then its genus $g = \frac{(m-1)(m-2)}{2} - d$ where d is a positive integer which is the measure of the smoothness of E. If E has only ordinary double points, d is simply the number of singular points. In particular the genus of a plane smooth projective curve is: $g = \frac{(m-1)(m-2)}{2}$. For a plane singular projective curve of degree m having multiple points P of multiplicity r_P in which it haver_P distincts tangents the genus is calculated as $g = \frac{(m-1)(m-2)}{2} - \sum_P \frac{r_P(r_P-1)}{2}$.

Corollary 13: (The Mordell conjecture [75]) Let E an algebraic curve defined on $\mathbb{Q}$ of genus $g \geq 2$, then E ($\mathbb{Q}$) is finite.

Remark: Consider the equation $P(x, y) = 0$ with P a polynomial having rational coefficients. The problem is the find the number N of solutions of this equation in Q. N depends of the genus g of the curve C associated to this equation (we can define the genus of C as the number of possibilities to cup the cuve without having two distinct parts):

*If: g=0: N=0 or N=∞.

*if: g=1: N=0 or C is elliptic.

*if: g≥ 2: Mordell conjectured that there are only finitely many solutions.

Example: The curve $y^2 = x^5 + x + 1$ in the plane has genus 2. The Mordell conjecture says that: $x^5 + x + 1$ is a square of a rational number for only finitely many rational values of x.

Remark: (1) Louis Joel Mordell (1888-1972) is a British mathematician.

(2)This conjecture was announced by Mordell in 1922 (see [75]).

(3) This conjecture was proved in general by Gerd Faltings 1983(see [33]).

(4) Vojta gives a proof a long of the lines of Diophantine approximation (see [110]).

Proof: (of corollary13)

For the deduction of the Mordell conjecture, from the abc-conjecture, see Machiel Van Frankenhuysen [36] and Emeline Crouseilles and Alexandre Lardeur [17].

The principle of the proof is as follows:

1. we construct a Belyi function having certain particularities using the Belyi theorem:

Theorem 3:(Belyi theorem [5]): if C is an algebraic curve defined on $\mathbb{Q}$, and if Σ is a subset of algebraic points of C, then it exists an application $f: C \to \mathbb{P}^1$ defined on $\mathbb{Q}$ associated to Σ such that f is uniquely ramified on 0, 1, ∞ and $f(\Sigma) \subseteq \{0,1,\infty\}$.

2. *we use the abc-conjecture to prove that we have two cases:

First case:$x \in C(\mathbb{Q})$ is sanded by f on 0, 1,∞.

Or

Second case: The height of $f(x)$ is bounded by an explicit constant.

*Recall that:

****A valuation:**** on $\mathbb{Q}$ Is an application $v: \mathbb{Q} \to \mathbb{R} \cup \{-\infty\}$ such that:

(i)$v(x) = -\infty \Leftrightarrow x = 0$.

(ii)$\forall x, y \in \mathbb{Q}^*\ v(xy) = v(x) + v(y)$.

(iii)$\exists K$ a constant such that $\forall x, y \in \mathbb{Q}$: $v(x + y) \leq K + \max\big(v(x), v(y)\big)$.

****Valuation p-adic on**** $\mathbb{Q}$: $v_p(x) = -ord_p(x)\ln(p)$ and $v_\infty(x) = \ln(x)$.

Where, if p is prime, $ord_p(x)$ is the power of the factor p in x.

We have: $\begin{cases} v_p(x+y) \leq \max\left(v_p(x), v_p(y)\right) \\ v_\infty(x+y) \leq \ln(2) + \max\left(v_\infty(x), v_\infty(y)\right) \end{cases}$.

The trivial valuation is defined by: $\begin{cases} v(0) = -\infty \\ v(x) = 0\ if\ x \neq 0 \end{cases}$.

The trivial valuation and the p-adic valuations represent the set V of valuations on $\mathbb{Q}$.

We have: $\sum_{v\in V} v(x) = 0$.

<u>The Logarithmic height of a point:</u>** for a point $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q})$, we define its logarithmic height by: $h(x) = \sum_{v\in V} \max\big(v(x_0) : \ldots : v(x_n)\big)$.

We have: $\forall C > 0$ *the set* $\{x = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{Q}), h(x) \leq C\}$ is finite.

We use the below form of the ABC conjecture in $\mathbb{P}^2$:

<u>abc-conjecture in $\mathbb{P}^2$</u>**:$\forall \epsilon > 0 \exists K(\epsilon) > 0$ such that: $\forall P = (a : b : c) \in \mathbb{P}^2$ lying on the line $a + b = c$ with $abc \neq 0$, we: have:$\max(h(P) - r(P), 0) \leq \epsilon h(P) + K(\epsilon)$.

Where: $h(p) = h(a : b : c) = \sum_v \max\,(v(a), v(b), v(c))$ v running over all valuations of $\mathbb{Q}$ and$r(p) = r(a : b : c) = \sum_{\text{p such that card}\{v_p(a), v_p(b), v_p(c)\} \geq 2} \ln\,(p)$, $card(A)$ denoting the number of elements of the set A.

10. THE SZAPIRO CONJECURES:

<u>Remark:</u> Lucien Szpiro is a French Mathematician born in 1941 and dead in 2020.

<u>Definition12</u>: In mathematics, we call cusp, a particular type of singular points on a curve. In the case of a curve having the equation:$f(x, y) = 0$, the cusps havent the below properties:

1. $f(x, y) = 0$
2. $\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$;
3. The hessienne matrix (composed by the second derivatives) has a null determinant.

<u>Definition13</u> :(1) (i) a cubic curve E on $\mathbb{Q}$ is an equation of the form:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

Where $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q}$ are not uniquely determined by the curve.

(ii) The correspondent form associated to a possible value of $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q}$ is called a model of E.

(2) This curve is called the Weierstrass normal equation.

(3)(i) If:$\begin{cases} b_1 = a_1 a_3 + 2a_4 \\ b_2 = a_1^2 + 4a_2 \\ b_6 = a_3^2 + 4a_6 \\ b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2 \end{cases}$ the number $\Delta(E) = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6$ is called the discriminate of the curve E.

(ii) Δ depends of the form of the equation of E.

(4) The below assertions are equivalent:

(i) E is elliptic.

(ii) E is liss.

(iii) The function A(x) =$x^3 + a_2 x^2 + a_4 x + a_6$ has not multiples roots on $\overline{\mathbb{Q}}$ (the set of algebraic numbers).

(iv) We can define a tangent to the curve in any point $(x, y) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$.

(v) The normal Weierstrass equation is not singular.

(vi) The normal Weierstrass equation is without any cusp and without any double point.

Example: 0 is a cusp of the equation $y^2 = x^3$ and is a double point of the equation $y^2 = x^2(x+1)$.

(vii)$\Delta(E) \neq 0$.

(viii) The curve $P(x,y) = y^2 + a_1xy + a_3y - x^3 - a_1x^2 - a_4x - a_6$ is liss.

(ix) $\forall(x,y) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$: The vector $(\frac{\partial P}{\partial x}(x,y), \frac{\partial P}{\partial y}(x,y)) \neq (0,0)$.

(x) $\forall(x,y) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$: $a_1y - 3x^2 - 2a_1x - a_4 \neq 0$ or $2y - a_1x + a_3 \neq 0$.

Remark: (1) The curve E ($\mathbb{Q}$): $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ gives, via the bijection:$(x,y) \to (X,Y) = \left(x, y + \frac{a_1}{2}x + \frac{a_3}{2}\right)$, the curve $E'(\mathbb{Q})$: $Y^2 = X^3 + \frac{b_2}{4}X^2 + \frac{b_4}{2}X + \frac{b_6}{4}$ with $\begin{cases} b_2 = 4a_2 + a_1^2 \\ b_4 = 2a_4 + a_1a_3. \\ b_6 = 4a_6 + a_3^2 \end{cases}$

We have: E ($\mathbb{Q}$) elliptic $\Leftrightarrow$ $E'(\mathbb{Q})$elliptic.

(2) The curve E ($\mathbb{Q}$): $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ gives, via the bijection: $(x,y) \to (X = x + \frac{a_1^2+4a_2}{12}, Y = y + \frac{1}{2}(a_1x + a_3))$ gives the curve $E''(\mathbb{Q})$: $Y^2 = X^3 + c_4X + c_6$ with:

$$\begin{cases} c_4 = a_4 + \dfrac{a_1a_3}{2} - 3(\dfrac{a_1^2 + 4a_2}{12})^2 \\ c_6 = a_6 + \dfrac{a_1^2}{4} - (\dfrac{a_1^2 + 4a_2}{12})^3 - \dfrac{a_1^2 + 4a_2}{12}(a_4 + \dfrac{a_1a_3}{2} - 3(\dfrac{a_1^2 + 4a_2}{12})^2) \end{cases}$$

(6) (i) For an elliptic curve: the number $J(E) = \frac{(b_2^2 - 24b_4)^3}{\Delta}$ is called the j-invariant of E.

(ii) $J(E)$is independent of the form of the equation of E.

(7) An elliptic curve has 3 “natural and interesting “different models:

(i) A model defined by a Weierstrass equation of the form: $y^2 = x^3 + c_4x + c_6$, having the discriminate $\Delta(E) = -16(4{c_4}^3 + 27{c_6}^2)$ with $c_4, c_6 \in \mathbb{Z}$.

(ii) The minimal model i.e. if $\Delta(E)$ is its discriminate, any other equation of E with integral coefficients will have a discriminate $\Delta = u^{12}\Delta(E)$ with $u \in \mathbb{Z}$.

(iii)The Néron model (For the definition and some properties see [124]).

(5)* Denote, for $p \in \mathbb{P}$ (a prime integer), by E_p the reduction, modulo p, of the elliptic curve E.

*So: in the prime field $\mathbb{Z}/p\mathbb{Z}$: E_p is defined by the equation $y^2 = x^3 + ax + b$, where: $\begin{cases} a, b \in \mathbb{Z} \\ p \in \mathbb{P}\ (p \text{ is prime}) \\ 0 < |a| < p \\ 0 < |b| < p \end{cases}$.

*We say that:

(i)That E has a good reduction in p if E_p is liss (i.e. p das not divide $\Delta(E) =$ the discriminate of the original curve).

(ii) That E has a bad reduction in p if E_p is not liss (i.e. p divides $\Delta(E)$).

(iii) That E has an additive reduction in p if E_p has a knot (i.e. if this model is singular modulo p and the singularity has a single tangent) (i.e. p divides c_4 and$\Delta(E)$).

(iv) That E has a multiplicative reduction in p if E_p has a pointe (i.e. if this model is singular modulo p with two distinct tangents) (i.e. p divide $\Delta(E)$ and das not divide c_4):

*if the two tangents are defined on the finite Field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ we say that the multiplicative reduction is deployed.

*if the two tangents are not defined on $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ we say that the multiplicative reduction is not deployed.

Remark: if we reduce the coefficient of the elliptic curve E modulo p we obtain an elliptic curve E_p defined on the Field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The group E_p=E ($\mathbb{F}_p$) is a cyclic group isomorphic to $\mathbb{Z}/r\mathbb{Z}$ (with r=cardinal (E_p)) or is the product of two cyclic groups isomorphic to $\mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ (with: $r$dividing q and r dividing: $p-1$).

(6)(i) The conductor of the elliptic curve E is defined by $N(E) = \prod_{p \in \mathbb{P}} p^{n(E,p)}$.

Where: $n(E,p) = \begin{cases} 0 \text{ if } E \text{ has a good reduction in } p \\ 1 \text{ if } E \text{ has multiplicative reduction in } p \\ 2 + \delta(E,p) \text{if } E \text{ has an additive reduction in } p \end{cases}$ with: $\delta(E,p) = \begin{cases} 0 \ if\ p \geq 5 \\ \leq 8 \ if\ p = 2. \\ \leq 5 \ if\ p = 3 \end{cases}$

For example for E ($\mathbb{Q}$): $\mathrm{y}^2 + \mathrm{y} = \mathrm{x}^3 - \mathrm{x}^2 + 2\mathrm{x} - 2$, the primes of bad reduction are p=5 and 7. The reduction at p=5 is additive, while the reduction at p=7 is multiplicative. Hence N (E) =25×7=175.

(ii) We say that E is semi-stable in the prime p if $n(p,E) = 0$ or 1.

(iii) We say that E is semi-stable if it is semi-stable in any prime number p i.e. if its conductor is without square factor.

Example: Let $(a,b,c) \in \mathbb{Z}_*^3 such\ that \begin{cases} \gcd(a,b,c) = 1 \\ a+b+c = 0 \\ a \equiv -1\ [modulo\ 4] \\ b \equiv 0\ [modulo 16] \end{cases}$ the elliptic curve defined on $\mathbb{Q}$ having the equation $y^2 = (x+b)(x-a)x$ isomorphic to the elliptic curve having the equation $y^2 = (x-e_1)(x-e_2)(x-e_3)$ with $\begin{cases} a = e_2 - e_3 \\ b = e_3 - e_1 \\ c = e_1 - e_2 \end{cases}$ (so: E is invariant if a,b,c are permuted circularly), giving by the variable change $\begin{cases} x = 4X \\ y = 8Y + 4X \end{cases}$ the new Weierstrass equation $Y^2 + XY = X^3 + \frac{b-a-1}{4}X^2 - \frac{ab}{16}X$ with integral coefficients, has the minimal discriminate $\Delta(E) = (\frac{abc}{16})^2$, has the conductor $f(E) = r\left(\frac{abc}{16}\right) = \prod_{p\in\mathbb{P}, p\ dividing\ \frac{abc}{16}} p$ and has the invariants $c_4 = -(ab + ac + bc)$ and $c_6 = \frac{(b-a)(c-b)(a-c)}{2}$.

Remark: (1) gcd $(c_4, \Delta) = 1$ ⇒the new equation defines a minimal model of E, so E is an elliptic curve semi-stable of good reduction in a prime integer l if l das not divide $\frac{abc}{16}$ and has bad reduction of multiplicative type if l divides $\frac{abc}{16}$.

(2) If we have not: $\begin{cases} a \equiv -1[\text{modulo}4], b \equiv 0[\text{modulo}16] \\ b \equiv -1[\text{modulo}4], c \equiv 0[\text{modulo}16] \\ c \equiv -1[\text{modulo}4], a \equiv 0[\text{modulo}16] \end{cases}$ ⇒ the equation $y^2 = (x + b)(x - a)x$ is minimal ⇒the curve is not semi-stable in p=2.

Corollary 14: (Szpiro conjecture [107], [108]) (announced in 1980): $\forall\epsilon > 0 \exists C(\epsilon)$ such that $\forall E$ an elliptic curve defined on $\mathbb{Q}$ with minimal discriminate Δ and conductor f, we have: $|\Delta| \leq C(\epsilon) f^{6+\epsilon}$.

Proof: (of corollary 14)

For a proof see A.Nitaj [79], [80].

Corollary 15: (The Generalized Szpiro conjecture [107], [108]): $\forall\epsilon > 0 \forall M > 0 \exists K(\epsilon, M) > 0$ such that: $\forall x, y \in \mathbb{Z}$ $D = 4x^3 - 27y^2 \neq 0$ and the greatest prime factor of x, y is bounded by $M \Rightarrow \max(|x|^3, y^2, D) < K(\epsilon, M) r(D)^{6+\epsilon}$.

Corollary 16: (The modified Szpiro conjecture [107], [108]) $\forall\epsilon > 0 \exists C(\epsilon) > 0$ such that $\forall E$ an elliptic curve defined on $\mathbb{Q}$ with invariants C4, C6 and conductor f, we have:

$$\max(|C_4|^3, |C_6|^2) \leq C(\epsilon) f^{6+\epsilon}$$

Corollary 17: (Lang's [59], [60] conjecture giving a lower bound for the height of a non-torsion rational point of an elliptic curve) let E be an elliptic curve defined on $\mathbb{Q}$, $E(\mathbb{Q})$ be the additive group of all the rational points on the curve E. For q $E(\mathbb{Q})$, the canonical height of q is:$\bar{h}(q) = \lim_{n\to+\infty} \frac{h(2^n q)}{4^n}$, where $h(q) = h(x(q))$ is the logarithmic height of q, defined by: $h\left(\frac{s}{t}\right) = \ln(\max(|s|, |t|))$ for $\frac{s}{t} \in \mathbb{Q}$ in the lowest forms is the logarithmic height of s/t: the Lang conjecture says that: for any elliptic curve, and any q $\mathbb{Q}$ of infinite order, we have:

$$\bar{h}(q) \geq C1 \ln(\Delta(E)) + C2$$

With: $\Delta(E)$ is the discriminate of the curve and C1, C2 are absolute constants.

Remark: (1) Serge Lang (1927-2005) is a French-American Mathematician.

(2)See Silverman 2010 [99] for more details about Lang's conjecture.

11. Cochrane/Dressler conjecture about the gaps between primes:

Corollary 18: (Cochrane-Dressler 1999 [16]) $\forall\epsilon > 0 \exists K(\epsilon) > 0$ such that $\forall(a, c)$ positive integers, we have:

$$\begin{cases} a < c \\ a, b \text{ have the same prime factors} \end{cases} \Rightarrow c - a \geq K(\epsilon) a^{\frac{1}{2}-\epsilon}$$

Remark: Cochrane and Dressler noted in [16] that the exponent: " $\frac{1}{2} - \epsilon$" cannot be improved to the exponent: "$\frac{1}{2}$". Indeed they found an infinite family of pairs of positive integers a<c having the same prime factors

such that: $c - a < \sqrt{2\ln(2)}\frac{a^{\frac{1}{2}}}{(\ln(a))^{\frac{1}{2}}}$, by considering the integers:a_1=$2(2^k-1)^2$and $c_1 = 2^{k+1}(2^k-1)$, (for k any positive integer) having the same prime factors and $c_1 - a_1$=$\sqrt{2}a_1^{\frac{1}{2}}$. Suppose that k=$23^{j-1}$ with j an integer ≥ 2. We have 3^j divides 2^k-1 and 3^{j-1} divides $c_1$and a_1. So, we obtain the smallest integers:$a = \frac{a_1}{3^{j-1}}$ and $c = \frac{c_1}{3^{j-1}}$ having the same prime factors and satisfying: $-a = \frac{\sqrt{2}a^{\frac{1}{2}}}{3^{\frac{j-1}{2}}} = \frac{2a^{\frac{1}{2}}}{\sqrt{k}}$. So:$c - a < \sqrt{2\ln(2)}\frac{a^{\frac{1}{2}}}{(\ln(a))^{\frac{1}{2}}}$.

Proof: (of corollary 18)

*Suppose that a<c are positive integers having the same prime factors. Rearranging we get:$b = c - a$.

*Let $P = r(a) = r(c)$ and $d = \gcd(a,b) = \gcd(a,c) = \gcd(b,c)$

*Then we have: $\frac{a}{d} + \frac{b}{d} = \frac{c}{d}$ with: $\gcd\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right) = 1$.

*We have: $r\left(\frac{a}{d}\frac{b}{d}\frac{c}{d}\right) \leq r(ac)r\left(\frac{b}{d}\right) \leq P\frac{b}{d} \leq \frac{b^{é}}{d}$.

(We have: $P \leq b$ because: P divides b since P divides a and P divides c imply P divides $b = c - a$).

*By the abc-conjecture, we have: $\frac{c}{d} \leq C(\epsilon)(\frac{b^2}{d})^{1+\epsilon}$.

*So: $c \leq C(\epsilon)b^{2(1+\epsilon)}$.

*Finally: $b \geq C'(\epsilon)c^{\frac{1}{2}-\epsilon} \geq C'(\epsilon)a^{\frac{1}{2}-\epsilon}$.

*As a consequence, we can deduce the below results:

****Corollary 19:** Between any two positive integers having the same prime factors there is a prime.

Cochrane and Dressler state in [16]:

****Corollary 20:** if the prime factors of a and c are restricted to a fixed finite set S of prime, then:

$$\exists K(S) > 0 \text{ such that } c - a \geq \frac{a}{(\ln(a))^{K(S)}}$$

12. SEPARATION BETWEEN PERFECT POWERS CONJECTURES:

(i) THE STRONG AND THE WEAK HALL'S CONJECTURES:

Corollary 21: (Strong Marshall Hall's conjecture [48])$\exists C > 0$ such that: $\forall x, y \in \mathbb{Z}^*$ $y^2 \neq x^3 \Rightarrow |y^2 - x^3| > C|x|^{\frac{1}{2}}$.

Remark:

1) Marshall Hall Jr. (1910-1990) is an American mathematician who made significant contributions to group theory and combinatorics.

2) Marshall announced his conjecture in 1970.

3) The conjecture arose from consideration of the Mordell equation in the theory of integers points on elliptic curves.

4) Marshall suggested that one could take: $C = \frac{1}{5}$. This is consistent with all the known data at 1970.

5) Noam Elkies showed [26], in 1998, that Hall's conjecture requires C being $<\frac{1}{50}$ (which is roughly 10 times smaller than $\frac{1}{5}$ suggested by Hall) by noting that:

$$447884928428402042307918^2 - 5853886516781223^3 = -1641843:$$

Corollary 22: (Weak form of Hall's conjecture) we have:

$$\forall \epsilon > 0 \exists C(\epsilon) > 0\ \forall x, y \in \mathbb{Z}^*\ y^2 \neq x^2 \Rightarrow |y^2 - x^3| > C(\epsilon)|x|^{\frac{1}{2}-\epsilon}$$

Remark: According to Wikipedia [131]: the weak form of Hall's conjecture was stated by Stark and Trotter around 1980.

Proof: (of corollary 22)

See Schmidt, Wolfgang M. (1996) [90] and Jerzy Browkin (2012) [15] (p 86) where he shows that the weak Hall conjecture is a consequence of a weak form of the abc-conjecture.

Remark: 1) In 1982 Danilov [20] showed that:"$|x|^{\frac{1}{2}}$" in the Marshall conjecture, cannot be replaced by: "$|x|^{\frac{1}{2}+\delta}$" for any $\delta > 0$.

2) Davenport [22] showed, in 1965, the below analogue of Hall's conjecture in the case of polynomials:

Proposition 17: (Hall's Conjecture for polynomials) if $f(t)$ and $g(t)$ are nonzero polynomials over **C**, then: $(g(t))^3 \neq (f(t))^2$ in $\mathbf{C}[t] \Rightarrow \deg\left(\left(g(t)\right)^2 - \left(f(t)\right)^3\right) \geq \frac{1}{2}\deg\left(f(t)\right) + 1$.

A generalization to other perfect powers is Pillai's conjecture.

(ii) THE PILLAI'S-TIJDEMAN CONJECTURES:

Corollary 23 :(Tijdeman generalized conjecture) [109] we have:

$$\forall k \in \mathbb{N}^* \text{ The set } T_k = \{(x, y, m, n) \in \mathbb{N}^* \times \mathbb{N}^* \times (\mathbb{N}^* - \{1\}) \times (\mathbb{N}^* - \{1\}), y^m = x^n + k\} \text{ is finite}$$

Remark: 1) Robert Tijdeman is a Dutch mathematician born in 1943.

2) Tijdeman theorem, proved in 1976, answers the case k = 1.

3) Tijdeman announced this conjecture in [109].

Corollary 24 :(Pillai's conjecture) [83] we have:

$$\forall A, B, k \in \mathbb{N}^* \text{ the set } P_{A,B,k} = \{(x, y, m, n)\mathbb{N}^* \times \mathbb{N}^* \times (\mathbb{N}^* - \{1\}) \times (\mathbb{N}^* - \{1\}), By^m = Ax^n + k\} \text{ is finite.}$$

Remark: 1) S. S. Pillai is an Indian mathematician born in 1901 and dead in 1950.

2) Pillai announced his conjecture in 1931[83].

3) The Pillai conjecture says that: "in the sequence of perfect powers the difference between two consecutive terms tends to infinity" or equivalently that: "letting k a positive integer, then the Diophantine equation $y^m - x^n = k$ in the integers: $x, y, n, m \geq 2$, has only finitely many solutions x, y, n, m".

(iii) THE LANG-WALDSCHMIDT CONJECTURE:

The abc-conjecture implies the following stronger versions of Pillai's conjecture called Lang-Waldschmidt conjecture:

Corollary 25: (Lang [58]-Waldschmidt [111], [112] conjecture) $\forall\epsilon > 0\ \exists K(\epsilon) > 0$ such that:

$$x^p \neq y^q \Rightarrow |x^p - y^q| \geq K(\epsilon)(\max(x^p, y^q))^{1-\frac{1}{p}-\frac{1}{q}-\epsilon}$$

Proof: (of corollary 25)

See in Serge Lang [59] the introduction of chapters X and XI.

Remark: 1) in the special case:$p = 3, q = 2$, the Lang-Waldschmidt conjecture reads:

$$x^3 \neq y^2 \Rightarrow |x^3 - y^2| \geq K(\epsilon)(\max(x^3, y^2))^{\frac{1}{6}-\epsilon}$$

2) in 1971, Marshall Hall Jr proposed the stronger conjecture below which does not follow from the abc-conjecture (See [36] and [49]):

Corollary 26 :(Strong Hall conjecture) $\exists K > 0$ an absolute constant such that:

$$x^3 \neq y^2 \Rightarrow |x^3 - y^2| \geq K(\max(x^3, y^2))^{\frac{1}{6}}$$

In [49] Marshall Hall discusses the values of K. In This sense L.V.Danilov (See [20], [52]) proved that:

Proposition 18: The inequality $0< |x^3 - y^2| < 0.971\sqrt{|x|}$ has infinitely many solutions in integers: x, y.

(iv) BEUKERS-STEWART CONJECTURE:

According to F. Beukers and C.L Stewart in [8]: "The M. Hall. Jr conjecture maybe too optimistic". They, then conjecture:

Corollary 27: (Beukers-Stewart conjecture [8]): Let p, q coprime integers with $p > q \geq 2$, then:$\forall K > 0$, there exist infinitely many positive integers x, y such that: $|x^p - y^q| < K(\max(x^p, y^q))^{1-\frac{1}{p}-\frac{1}{q}}$

Proposition 19: (Vojta, P. [110]) we have:

(a) The below conjectures are equivalent:

(1) The abc-conjecture.

(2) The Hall-Lang-Waldschmidt-Szpiro conjecture.

(3) Generalized Szpiro conjecture.

(b) The Hall-Lang-Waldschmidt-Szpiro conjecture ⇒The Hall-Lang-Waldschmidt conjecture⇒Hall conjecture.

(c) Generalized Szpiro conjecture⇒Szpiro conjecture⇒asymptotic Fermat.

13. FINITELY MANY PERFECT POWERS OF AN INTEGRAL POLYNOMIAL HAVING AT LEAST THREE SIMPLE ZEROS:

Corollary 28: The abc-conjecture implies that for an integral polynomial P(x) we have:

P has at least three simple zeros⇒ P has only finitely many perfect powers for all integers x. [120]

14. BOUNDING ABOVE C BY A NEAR-LINEAR FUNCTION OF THE RADICAL OF ABC:

Corollary 29: The abc-conjecture implies that c can be bounded above by a near-linear function of the radical of abc. Bounds are known that are exponential. Specifically, the following bounds have been proven:

(1) **Stewart and Tijdeman inequality (See [100]):** $c < e^{K_1(r(abc))^{15}}$

(2) **Stewart and Yu first inequality(See [101]):** $c < e^{K_2(r(abc))^{\frac{2}{3}+\epsilon}}$

(3) **Stewart and Yu second inequality (See[102])** :$c < e^{K_3(r(abc))^{\frac{1}{3}+\epsilon}}$

REMARK: In these bounds, $K1, K2$ and $K3$ are constants that depend on ϵ (in an effectively computable way) but not on $a, b,$ or c. The bounds apply to any triple for which $c > 2$.

15. LAISHRAM-SHOREY RESOLUTION OF THE NAGELL-LJUNGGREN EQUATION VIA THE ABC-CONJECTURE:

Definition14: (the Nagell [77]-Ljunggren [61] equation) we call the Nagell-Ljunggren equation, the equation: $y^q = \frac{x^n-1}{x-1}$ in integers $x > 1, y > 1, n > 2$ *and* $q > 1$.

Remark: in the basis x all the bits of the perfect power y^q are equal to 1.

Corollary 30: (Laishram-Shorey [56]) the abc-conjecture implies that the single solutions of the Nagell-Ljunggren equation are:

$$11^2 = \frac{3^5 - 1}{3 - 1}\ ; 20^2 = \frac{7^4 - 1}{7 - 1} \text{ and } 7^3 = \frac{18^3 - 1}{18 - 1}$$

Proof: (of corollary 30)

*We reproduce here the Laishram-Shorey proof [56].

*Proceed by the absurd reasonning and Let $x > 1, y > 1, n > 2$ *and* $q > 1$be a non-exceptional solution of the Diophantine equation:$y^q = \frac{x^n-1}{x-1}$.

***Claim28:** There are no further solutions of $y^q = \frac{x^n-1}{x-1}$ for $q = 2$.

Proof: (of claim28)

For a proof see Ljunggren [61].

*So, we may suppose that $q \geq 3$.

***Caim29:** we have: 4 do not divide n.

Proof: (of claim 29)

For a proof see Nagell [77].

***Claim30:** we have: 3 do not divide n.

Proof: (of claim30)

For a proof see Ljunggren [61].

***Claim31:** We have: 5 and 7 do not divide n.

Proof: (Of claim 31)

For a proof see Bugeaud, Hanrot and Mignotte [15].

*So: $n \geq 11$.

*We have: $y^q = \frac{x^n - 1}{x-1} \Leftrightarrow x^n = (x-1)y^q + 1 \text{ and } q \geq 3 \Rightarrow y < x^{\frac{n}{q}} \leq x^{\frac{n}{3}}$ and $r(x(x-1)y) < x^2 y < x^{2+\frac{n}{3}}$.

*But by the Baker version of the abc-conjecture, we have:

$$\begin{cases} x^n = (x-1)y^q + 1 \\ \gcd(x, y) = 1 \end{cases} \Rightarrow x^n < (r(x(x-1)y))^{\frac{7}{4}} < x^{\frac{7}{4}(2+\frac{n}{3})} \Rightarrow n < \frac{7}{2} + \frac{7n}{12} \Rightarrow n \leq 8$$

Conclusion: Having obtained the impossible relation$11 \leq n \leq 8$: our starting absurd hypothesis "$\exists x > 1, y > 1, n > 2 \text{ and } q > 1$ a non-exceptional solution of the Diophantine equation $y^q = \frac{x^n - 1}{x-1}$" is not true.

16. LAISHRAM-SHOREY RESOLUTION OF THE DIOPHANTINE EQUATION $n(n+d)\dots(n+(k-1)d) = by^l$

Consider the Diophantine equation $n(n+d)\dots(n+(k-1)d) = by^l$

Corollary 31: (Granville [57]) the abc-conjecture and $n(n+d)\dots(n+(k-1)d) = by^l$ and $l = 2,3 \Rightarrow k$ is bounded by an absolute constant.

Proof: (of corollary 31)

See Laishram (2004) [57], p 69.

Corollary 32: (Shorey [94]) the abc-conjecture and $n(n+d)\dots(n+(k-1)d) = by^l$ and $l \geq 4 \Rightarrow k$ is bounded by an absolute constant.

Proof: (of Corollary 32)

See Shorey (1999) [94].

Corollary 33: (Gyory-Hajdu-Saradha [47]) the abc-conjecture implies that for a given $k \geq 3$: the Diophantine equation $n(n+d)\dots(n+(k-1)d) = by^l$has only finitely many solutions in positive integers $n, d > 1, b, y$ and $l \geq 4$.

Proof: (of corollary 33)

For a proof see Gyory-Hajdu-Saradha [48].

Corollary 34: (Saradha [89]) we have:

(1)$\begin{cases} n(n+d)\dots(n+(k-1)d) = by^l \\ k \geq 8 \end{cases} \Rightarrow l \leq 29$.

(2)$l = 29 \Rightarrow k \leq 8$.

(3)$l \in \{19,23\} \Rightarrow k \leq 32$.

(4) $l = 17 \Rightarrow k \leq 10^2$.

 ISSN 2349-0292
Impact Factor 3.802

(5) $l = 13 \Rightarrow k \leq 10.^7$

(6) $l \in \{7{,}11\} \Rightarrow k \leq e^{e^{280}}$.

Proof: (of corollary 34)

For a proof see Saradha [89].

Corollary 35: $(k,l) \in \{(3{,}2);(3{,}3);(4{,}2)\} \Rightarrow$ the Diophantine equation: $n(n+d)\ldots(n+(k-1)d) = by^l$ has infinitely many solutions.

Conjecture: it is conjectured that:

$$\exists n, d > 1, y \geq 1, b, l \geq 2 \text{ and } k \geq 3 \text{such that} \begin{cases} n(n+d)\ldots(n+(k-1)d) = by^l \\ \gcd(n,d) = 1 \\ P(b) = \text{the greatest prime factor } of\ b \leq k \end{cases} \Rightarrow (k,l) \in \{(3{,}2),(3{,}3),(4{,}2)\}$$

Remark: for an account on the Diophantine equation $n(n+d)\ldots(n+(k-1)d) = by^l$ see Shorey (2002) [95], [96] and Shorey-Saradha 2005 [98].

Corollary 36: The abc-conjecture implies that:

$$(1)\begin{cases} n(n+d)\ldots(n+(k-1)d) = by^l \\ n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1, l > 1 \\ \gcd(n,d) = 1 \\ P(b) = \text{the greatest prime factor of } b \leq k \end{cases} \Rightarrow l \leq 7.$$

(2) If $l = 7$, we have: $k < e^{13006.2}$.

Proof: (of claim36)

For a proof see Laishram-Shorey 2011 [56].

17. THE GOORMAGHTIGH EQUATION:

Definition15: (Goormaghtigh equation [44]) the Goormaghtigh equation is the Diophantine equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$ in $x > 1, y > 1, m > 2, n > 2$ with $x \neq y$.

Remark :1) René Goormaghtigh (1893-1960,) was a Belgian engineer, after whom the Goormaghtigh conjecture is named.

2) We can assume without loss of generality that: $x > y > 1$ and $n > m > 2$

3) See Shorey 1999 [94] for a survey of results on the Goormaghtigh equation.

Weak conjecture: There are only finitely many solutions x, y, m, n of the Goormaghtigh equation.

Strong Conjecture: It is conjectured that: $31=\frac{5^3-1}{5-1} = \frac{2^5-1}{2-1}$ and $8191=\frac{90^3-1}{90-1} = \frac{2^{13}-1}{2-1}$ are the sole solutions of the Goormaghtigh equation.

Corollary 37: (Laishram-Shorey [56]) the abc-conjecture implies that the Goormaghtigh equation has only finitely many solutions in integers: $x > 1, y > 1, m > 2, n > 3$ with $x \neq y$. More precisely we have:

(1) $\begin{cases} \frac{x^m-1}{x-1} = \frac{y^n-1}{y-1} \\ x >, y > 1, m > 2, n > 3, x > y \end{cases} \Rightarrow m \leq 6$

(2) $m = 6 \Rightarrow 7 \leq n \leq 17$ $with$ $n \notin \{11,16\}$

(3) $\exists C > 0$ an effectively computable absolute constant such that: max $(x, y, n) \leq C$

Remark: Laishram and Shorey say [56] that "This improves considerably theorem 1.4 of Saradha [88]"

Proof: (of Corollary 37)

For a proof see Laishram-Shorey 2011 [56].

18. EDWARD WARING PROBLEM:

Definition16: (Waring problem [113]) this problem asks whether it is true that: $\forall k \in \mathbb{N}\, \exists s \in \mathbb{N} \forall N \in \mathbb{N} \exists$ at most s natural numbers $x_1, x_2, \dots, x_s$ such that:$N = \sum_{i=1}^{s} x_i^k$?

Remark : 1) the Waring problem was proposed in 1770 by the British Mathematician Edward Waring (1736-1798), after whom it is named. Waring wrote in [113]: « Omnis integer numerus vel est cubus, vel duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &, usque ad novemdecim compositus, & sic deinceps ».

2) Before Waring, the Alexandrian mathematician Diophantus (AD 200-284) had asked, in his « Diophantus Arithmetica » (see the Latin Translation by Claude Gaspard Bachet de Méziriac published in 1621) : « is it true that : $\forall N \in \mathbb{N}^* \exists x_1, x_2, x_3, x_4 \in \mathbb{N}$ such that : $N = x_1^2 + x_2^2 + x_3^2 + x_4^2$?». The Diophantus problem is known as « Bachet conjecture », Bachet being the translator of « Diophantus Arithmetica » from Grec language to Latin language.

3) The French Mathematician Pierre de Fermant claimed in 1640 to have a proof of the Diophantus problem but he did not never publish this proof.

4) In 1770, Some Months after the Waring problem announced, the French Mathematician Lagrange, J.L. confirms the "Diophantus problem", by showing that:"any positive integer is a sum of at most 4 squares". The Lagrange result is known as:"Lagrange Four-square theorem".

5) In 1986, François Dress, R. Balasubramanian and Jean-Marc Deshouillers show in [9] that: "any positive integer is a sum of at most 18 bi-squares".

6) For more details about Waring Problem see [63], [106], [111], [112].

Theorem 4 : (Hilbert [53]-Waring [113] theorem) gives an affirmative answer to the Waring problem.

Proof : (of theorem 4)

For a proof see Hilbert 1909 [53].

Definition 17: (the Waring function$g$) $\forall k$ integer≥ 2:

$$g(k) = \min \{s \in \mathbb{N}^*, \forall N \in \mathbb{N}\ \exists x_1, x_2, \dots, x_s \in \mathbb{N} \text{ such that } N = \sum_{r=1}^{s} x_r^k\}$$

 ISSN 2349-0292
Impact Factor 3.802

Definition18: for k integer ≥ 2 define $I(k) = 2^k + E\left(\left(\frac{3}{2}\right)^k\right) - 2$, where $E(t)$ denotes the entire part of the real t.

Proposition20: (Euler J. A. theorem [111], [112])$\forall k$ integer ≥ 2 we have: $g(k) \geq I(k)$.

Remark: Johann Albrecht Euler (1734-1800) is a Swiss mathematician son of the famous Swiss Mathematician Leonhard Euler (1707-1783). He proved proposition 19 as reproduced here (See [111], [112]).

Proof: (of proposition 19)

*By the Euclidean division of: 3^k by 2^k, we have:$3^k = 2^k q + r$ with $0 < r < 2^k$ and $q = E\left(\left(\frac{3}{2}\right)^k\right)$=the integer part of $\left(\frac{3}{2}\right)^k$.

*Consider the integer $N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k$.

*$N < 3^k \Rightarrow$ any sum in k-powers of N does not contain 3^k.

*$N < q2^k$ ⇒there is at most $q - 1$ terms are of the form $2^k$and all the others are of the form 1^k.

*So, the number of terms is $\geq q - 1 + 2^k - 1 = I(k)$.

*That is: $g(k) \geq I(k)$.

Conjecture: (of Bretschneider, C.A. (see [11])): $\forall k$ integer ≥ 2 we have: $g(k) = 2^k + E\left(\left(\frac{3}{2}\right)^k\right) - 2$.

Remark: Carl Anton Bretschneider (1808-1878) is a German mathematician. He announced his conjecture on 1853.

Theorem5: (Pillai S. [84]-Dickson L.E. [23], [24]) we have:

$$3^k - 2^k E\left(\left(\frac{3}{2}\right)^k\right) \leq 2^k - E\left(\left(\frac{3}{2}\right)^k\right) - 2 \Rightarrow g(k) = 2^k + E\left(\left(\frac{3}{2}\right)^k\right) - 2$$

Remark: Pillai and Dickson showed theorem 5 independently on 1936.

Remark: 1) Stemmler showed in 1990 [100] that the condition $3^k - 2^k E\left(\left(\frac{3}{2}\right)^k\right) \leq 2^k - E\left(\left(\frac{3}{2}\right)^k\right) - 2$ is satisfied by $4 \leq k \leq 401\,600\,000$

2) Kubina-Wunderlich showed in 1990 [55] that the condition $3^k - 2^k E\left(\left(\frac{3}{2}\right)^k\right) \leq 2^k - E\left(\left(\frac{3}{2}\right)^k\right) - 2$ is satisfied by $4 \leq k \leq 471\,600\,000$.

Theorem6: (Kurt Mahler (1903-1988) [64]) the condition $3^k - 2^k E\left(\left(\frac{3}{2}\right)^k\right) \leq 2^k - E\left(\left(\frac{3}{2}\right)^k\right) - 2$ is true for any integer k great enough.

Corollary38: (Laishram S. [57], [58]) the abc-conjecture⇒ $\forall k$ integer ≥ 2 we have: $g(k) = 2^k + E\left(\left(\frac{3}{2}\right)^k\right) - 2$.

Proof: (of corollary 38)

*We reproduce below the proof of Laishram given in [55], [56].

*We write $3^k = 2^k$ q + u with 0 < u < 2^k and q = $E\left(\left(\frac{3}{2}\right)^k\right)$.

*By the Dickson-Pillai theorem (theorem 5), The Mahler theorem (theorem 6) and the Kubina-Wunderlich result, the ideal Waring's Theorem holds provided that the remainder u = $3^k - 2^k q$ satisfies the inequality: u ≤ 2^k − q – 3, satisfied for 3 ≤ k ≤ 471600000 as well as for sufficiently large k.

*Suppose k > 471600000

*Proceed by the absurd reasonning and suppose that: u ≥ 2^k − q – 2

* Let gcd (3^k , 2^k (q + 1)) = 3^v and set a = 3^{k-v} , c = 3^{-v} 2^k (q + 1) and b = c – a = 3^{-v} (2^k − u).

*We have: $\begin{cases} a + b = c \\ b = 3^{-v}\,(2^k - u) \leq 3^{-v}(q+3). \\ \gcd(a,b,c) = 1 \end{cases}$

*Then r=r (abc) = r ($3^{k-v} \cdot 3^{-v}2^k$ (q + 1) · b) ≤ 6b (q + 1) 3^{-v} ≤ 6(q + 1) (q + 3) 3^{-2v} .

First case : if r < e^{63727}.

*By the Baker version of the abc-conjecture, we have : $2^k \leq 2^k$ (q + 1) $3^{-v} < r^{\frac{7}{4}} < e^{\frac{7}{4}\times 63727}$.

* So : k < $\frac{63727\times 7}{4\ln(2)}$ < 160893.

*This is impossible since k > 471600000.

Second case : if r ≥ e^{63727}.

*By The remark following the Baker version of the abc-conjecture (see proposition 9), written for: = $\frac{1}{3}$, we have :

$$2^k\,(q+1)\,3^{-v} < \frac{6}{5\sqrt{2\pi\times 6460}}\left(\frac{(q+1)(q+3)}{3^{2v}}\right)^{\frac{4}{3}}$$

*So : $2^k < \frac{6^{\frac{7}{3}}}{5\sqrt{12920\pi}}\, q^{\frac{5}{3}}\left(1+\frac{3}{q}\right)^{\frac{5}{3}}$.

* Since $3^k > 2^k$ q, we have q < $\left(\frac{3}{2}\right)^k$ and also : 1 + $\frac{3}{q}$ < 2 since k ≥ 3.

*Finally : $2^k < \frac{6^{\frac{7}{3}}}{5\sqrt{12920\pi}}\, q^{\frac{5}{3}}\left(1+\frac{3}{q}\right)^{\frac{5}{3}} < \frac{6^{\frac{7}{3}}\times 2^{\frac{5}{3}}}{5\sqrt{12920\pi}}\left(\left(\frac{3}{2}\right)^{\frac{5}{3}}\right)^k$ =0. 20617…× $(1.96\,\ldots)^k < 2^k$.

*This is being impossible the result follows.

Hardy and Littlewood introduced [49], [50] the more important function G defined by:

Definition 19: (the Waring function)) $\forall k$ integer≥ 2:

$$G(k) = \min\{s \in \mathbb{N}^*, \exists C(k) > 0 \forall N \in \mathbb{N}, N \geq C(k)\ \exists x_1, x_2, \ldots, x_s \in \mathbb{N} \text{ such that } N = \sum_{r=1}^{s} x_r^k\}$$

Proposition 21: $\forall k$ integer ≥ 2 we have: $G(k) \leq g(k)$.

ISSN 2349-0292
Impact Factor 3.802

Proof: (of proposition 21)

The result follows by definition of g and G.

Proposition 22: We have:

(1)$G(k) \geq 2^{l+2} if\ k = 2^l, l \geq 2\ or\ k = 3 \times 2^l$.

(2)$G(k) \geq p^{l+1}, if\ p \in \mathbb{P}$ (The set of prime integers), $p > 2\ and\ k = p^l(p-1)$.

(3)$G(k) \geq \frac{p^{l+1}-1}{2}$, if $p \in \mathbb{P}, p > 2\ and\ k = \frac{p^l(p-1)}{2}$.

Proof: (of proposition 22)

*The results follow by consideration the structure of the unîtes group of the ring$\mathbb{Z}/n\mathbb{Z}$.

*For example the relation (3) follows from the fact that any power $k^e \equiv -1, 0\ or\ 1$ [modulop^{l+1}].

Theorem7: (See Pascal Boyer [10]) $\forall k$ integer ≥ 2 we have: $G(k) \geq k + 1$.

Proof: (of theorem7)

For a proof see Boyer [10].

Conjecture: In the absence of congruence restrictions, by a density argument, we must have: G (k) = k + 1.

Theorem8: (A.A. Karatsuba [54]) $\forall k$ integer ≥ 404 $G(k) < 2k(\ln(k) + \ln(\ln(k)) + 6)$.

Proof: (of theorem8)

For a proof see Karatsuba [54].

Remark: For More Applications and consequences of the abc-conjecture see [17], [52], [78], [93], [111] and [112].

REFERENCES

[1] ABC@Home: a grid computing system by the mathematics department at Leiden University together with the Dutch Kennislink science Institute. Available at http://abcathome.com/data/ (Accessed on January 26, 2022)
[2] Alborghetti, Gael (2011): TIPE. La repartition des nombres premiers. Available: http://math.univ-Lyon.fr/~calderon/Alborghetti/ (Available at: February 11, 2022)
[3] Baker, Alain (1998): Logarithmic forms and the ABC conjecture. In Györy, Kalman (Ed). Number theory. Diophantine, Computatinal and algebraic aspects. Proceedings of the international conference, Eger, Hungary, July 29-August 2, 1996. Berlin: de Gruyter, pp 37-44 Available at: https://zbmath.org/?format=complete&q=an:0973.11047/(Accessed on January 28, 2022)
[4] Baker, Alain (2004): Experiments on the abc-conjecture. Publ. Math. Debrecen, 65, pp 253-260.
[5] Belyĭ, Gennadiĭ Vladimirovich (1980): Galois extensions of a maximal cyclotomic field. Math. USSR Izv. **14** (2): 247–256. Translated by Neal Koblitz.
[6] Berend, D. - Osgood, C.F (1992): On the equation P(x) = n! and a question of Erdos, J. Number. Theory, Vol 42, pp 189-193.
[7] Berndt, Bruce. C- Galway, William. F (2000): On the Brocard-Ramanujan Diophantine equation n! +1=m^2. The Ramanujan Journal. 4, pp 41-42. Available at: https://faculty.math.illinois.edu/~berndt/articles/galway.pdf/ (Accessed on January 23, 2022)

[8] Beukers-Stewart: Neighbouring Powers. Available at: https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.180.8688rep=rep/8type=pdf/ (Accessed on January 27, 2022)
[9] Balasubramanian, R.-Deshouillers, J-M.-Dress, F. (1986): Problème de Waring pour les bicarrés 1, 2. C. R. Acad. Sci. Paris Sér. I Math 303, pp 85-88 and pp 161-163.
[10] Boyer, Pascal (2019): Le petit companion des nombres et de leurs applications. Paris. Calvage et Mounet p648. V equations Diophantiennes chap 1.5 (Problème de Waring pp 469-470).
[11] Bretschneider, C.A. (1853): Taflen fur die zerlegung der zahlen bis 4100 in Biquadrate. In Journal fur die Reine und angewandte Mathematik. 46, pp 1-23.
[12] Brocard, Henri (1876): Question 166. Nouv.Corres.math. Vol 2, p287.
[13] Brocard, Henri (1885) : Question 1532. Nouv. Ann. Math. Vol 4, p 391.
[14] Browkin, Jerzy (2012): Number Theory. Published by R.P Bambah, V.C. Dumir and R.J. Hans-Gill. p 86. Available at: https://books.google.co.ma/books?id=aiDyBwAAQBAJ&pg=PA86&lpg=PA86&dq=the+weak+hall+conjecture&source=bl&ots=T43pFKS4yC&sig=ACfU3U3ao8QxV5m3t4CajHquRiW38CcM7w&hl=fr&sa=X&ved=2ahUKEwiS4M3y_D1AhXBoVwKHYkEC_YQ6AF6BAgxEAM#v=onepage&q=the%20weak%20hall%20conjecture&f=false/ (Accessed on February 8, 2002)
[15] Bugeaud, Y. - Hanrot, G. -Mignotte, M. (2002):sur l'équation diophantienne $\frac{x^n-1}{x-1} = y^q$. Proc. London. Math.Soc. 84, pp 59-78.
[16] Cochrane, Todd- Dressler, Robert E. (1999): Gaps between integers with the same prime factors Mathematics of computations, Vol 68, N° 225, pp 395- 401.
[17] Crouseilles, Emeline-Lardeur, Alexandre (2018) : la conjecture abc et quelques unes de ses conséquences. Sous la direction de Bernard Le Stum. Université de Rennes. Available at: https://perso.univ-rennes1.fr/ (Accessed on February 1, 2022)
[18] Cushing, David- Pascoe, J.E. (2016): Powerful numbers and the abc-conjecture. Available at: http://arxiv.org/abs/math/1611.01192v1/ (Accessed on February 4, 2022)
[19] Dąbrowski, A (1996): On the Diophantine equation n! +A=y^2. Nieuw. Arch. Wisk. Vol 14, pp 321-324.
[20] Danilov, L.D. (1982): The Diophantine equation $x^3 - y^2 = k$ and Hall's conjecture. Math. Notes Acad. Sci. USSR, 32, pp 617-618.
[21] Darmon, Henri - Granville, Andrew (1994): On the equation $z^m = f(x,y) and\ ax^p + by^q = cz^r$. Preprint N°28, volume II, university of Georgia.
[22] Davenport, H. (1965): On $(f(t))^3 - (g(t))^2$, K. Norske. Vid. Selskabs. Forh (Trondheim), 38, pp 86-87.
[23] Dickson, L. E. (1936) : Proof of the ideal Waring theorem for exponents, Amer. J. of Math., to 58, pp. 521-529.
[24] Dickson, L. E. (1936). : Solution of Waring's problem, Amer. J. of Math., t. 58, 1936, p. 530-535.
[25] Dickson, L. E (2005): Waring's Problem and related Results. Ch 25 in History of the theory of numbers, Vol 2. Diophantine analysis, New York, pp 717-729.
[26] Elkies, Noam D. (1998): Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction. Available at: http://arxiv.org/abs/math/0005139/ (Accessed on January 26, 2022)
[27] Epstein, Andew-Glickman, Jacob (2020) : Brocard Problem up to 10^{15}. Available at : https://github.com/jhg023/brocard/ (Accessed on February 9, 2022)
[28] Erdős, Paul- Oblath, R. (1937): Uber Diophantische Gleichungen der Form n! =$x^p \pm y^p$ und n! +m! =x^p, Acta Szeged, Vol 8, pp 241-255.
[29] Erdős, Paul (1965): Some recent advances and current problems in number theory. In lectures on modern mathematics III (edited by Saaty) John Wiley and Sons, pp 196-244.
[30] Erdős, Paul (1980): *How many pairs of products of consecutive integers have the same prime factors? (Research problem)*, Amer. Math. Monthly, vol. 87, pp. 391-392. Available at :

https://archive.wikiwix.com/cache/index2.php?url=http%3A%2F%2Fwww.renyi.hu%2F%7Ep_erdos%2F1980-11.pdf%2Findex.html#federation=archive.wikiwix.com/ (Accessed on January 29, 2022)

[31] Euclid (1966): les éléments T1 (livres I-VII), T2 (livres VIII-X), T3 (livres XI-XIII). In les œuvres d'Euclide. French Translation of the Euclid Grec work by F. Peyard. Blanchard. Paris. Available at: https://archive.org/detals/lesoevresd'euclid03eucl/ (Accessed on February 2, 2022)

[32] Euler, L (1770): Vostandige Anlectung Zur Algebra. Available at: https://rotichitropark.Firebaseapp.com/1421204266.pdf/(Accessed on January 20, 2022)

[33] *Faltings, Gerd (1983): Endlichkeitssätze für abelsche Varietäten über Zahlkörpern [Finiteness theorems for Abelian varieties over number fields]. Inventiones Mathematicae (in German).* ***73*** *(3): 349–366.*

[34] *Faltings, Gerd (1984) : Erratum: Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. Inventiones Mathematicae (in German).* ***75*** *(2): 381.*

[35] *Faltings, Gerd (1991) : Diophantine approximation on Abelian varieties. Ann. of Math.* ***133*** *(3): 549–576.*

[36] *Faltings, Gerd (1994) : The general case of S. Lang's conjecture. In Cristante, Valentino; Messing, William (eds.). Barsotti Symposium in Algebraic Geometry. Papers from the symposium held in Abano Terme, June 24–27, 1991. Perspectives in Mathematics. San Diego, CA: Academic Press, Inc.*

[37] *Frankenhuysen, Machiel Van: The abc-conjecture implies Roth's theorem and Mordell's conjecture. Available at:* http://mat.unb.br/16_4.pdf/*(Accessed on January 27, 2022)*

[38] Gauss, C.F: Neue Theorie de zeregung der cuben, In C.F Gauss Werke, Vol II, pp 383-390. Available at: http://archive.wikiwix.com/cache/index2.php?url=http%3A%2F%2Fresolver.sub.uni-goettingen.de%2Fpurl%3FPPN235957348/ (Accessed on: January 25, 2022)

[39] Gérardin, A (1906): Contribution à l'étude de l'équation 1.2.3.4…z+1=y^2, Nouv. Ann. Math, (4) 6, pp 222-226.

[40] Ghanim, Mohammed (10/12/2018): confirmation of the Fermat last theorem by an elementary short proof. GJAETS. India, the December 2018 issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2018/December-2018/1.pdf/ (Accessed on February 1, 2022)

[41] Ghanim, Mohammed (20/11/2019): Confirmation of the Beal-Brun-Tijdeman-Zagier conjecture. GJAETS. (India). The November 2019 issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2019/November-2019/1.pdf/ (Accessed on February 1, 2022)

[42] Ghanim, Mohammed (25/02/2021): A short elementary proof of the Beal conjecture with deduction of the Fermat last theorem. GJAETS. (India). The February 2021 issue. Available at: http://www.gjaets.com/Issues%20PDF/Archive-2019/February-2021/1.pdf/ (Accessed on January 25, 2022)

[43] Goldfeld, Dorian (1996): Beyond the last theorem. The sciences, Vol 36, issue 2. Available at: www.ega.math.norod.ru/ Available also at: https://doi.org/10.1002/j.2326-1951.1995.tb03243.x/ (Accessed on January 24, 2022)

[44] Goormaghtigh, René (1917) : L'Intermédiaire des Mathématiciens 24, p 88.

[45] Granville, Andrew- Tucker, Thomas (2002): It's easy as abc. Notices of the AMS. 49(10), pp 1224-1231 Available at: http://www.ams.org/notices/200210/fea-granville.pdf/(Accessed on January 28, 2022)

[46] Gupta, H. (1935): On a Brocard-Ramanujan problem, Math. Student, Vol 3, p71.

[47] Guy, Richard K. (1994): Unsolved problems in Number theory, Springer Verlag, New York.

[48] Györy, K.-hajdu, L.-Saradha, N. (2004): On the Diophantine equation: $n(n+d)\dots(n+(k-1)d)=by^l$, Cand. Math. Bull. 47, pp 373-388.

[49] Hall, Jr., Marshall (1971): The Diophantine equation $x^3 - y^2 = k$. In Atkin, A.O.L.; Birch, B.J. (Eds) Computers in Number theory, Proc. Atlas Sympos. N°2, Oxford 1969, pp 173-198.

[50] Hardy, G. H.-Littlewood, J. E. (1921-1928) : Some problems of "Partitio numerorum", I, Göttingen. Nachr., 1920, p. 33-54 ; II, Math. Z., t. 9, 1921, p. 14-27 ; IV, Math. Z., t. 12, 1922, p. 161-188 ; VI, Math. Z., t. 23, 1925, p. 1-37 ; VIII, Proc. London math. Soc., Series 2, t. 28, 1928, p. 518-542.

[51] Hardy, G. H.-Littlewood, J. E. (1919): A new solution of Waring' s problem, Quart. J. of pure and appl. Math., t. 48, p. 272-293.
[52] G. H. Hardy- W. M. Wright (1954) : An introduction to the theory of numbers, Oxford Univ. Press, third ed.
[53] Hilbert, David (1909): *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^ter Potenzen (Waringsches Problem)*, Math. Ann. vol. 67, pp. 281-300. Available at: https://archive.wikiwix.com/cache/index2.php?url=http%3A%2F%2Fgdz.sub.uni-goettingen.de%2Fdms%2Fload%2Fimg%2F%3FPPN%3DGDZPPN002262746%26IDDOC%3D38315#federation=archive.wikiwix.com/ (Accessed on January 31, 2022)
[54] Karatsuba, A.A. (1985): On the function G (n) in Waring's Problem. Izv. Akad. Nauk. SSSR Ser. Math, Vol 49, n°5, pp 935-947.
[55] Kubina, J.M.-Wunderlich, M.C. (1990): Extending Waring's conjecture to 471 600000. Math. Comput. 55, pp 815-820.
[56] Laishram, Shanta-Shorey, Tarlok (2011): Baker's explicit abc-conjecture and applications. Available at: https://arxiv.org/abs/math/1112.2461v1/ (Accessed on January 27, 2022)
[57] Laishram, S. (2004): Topics in Diophantine equations, M. Sc. Thesis, TIFR/Mumbai University. Available at: http://www.isid.ac.in/~shanta/MScThesis.pdf/ (Accessed on January 29, 2022)
[58] Laishram, S. (2015): Baker's explicit abc-conjecture and Waring's problem. Available at: https://www.isid.ac.in/~statmath/2015/isid201519.pdf/ (Accessed on February 2, 2022)
[59] Lang, Serge (1978): Elliptic curves. Diophantine analysis. Grundlehren der Mathematischen Wissenschaften, 231, Springer-Verlag. Berlin.
[60] Lang, Serge (1991): Number Theory III. Encyclopedia. Math. Sci 60, Springer.
[61] Ljunggren, W. (1943): Noen Setninger Om Ubestemte Likninger av formen $\frac{x^n-1}{x-1} = y^q$. Norsk. Mat. Tidsskr. 1, Hefte 25, pp 17-20.
[62] Luca, Florian (2002): The Diophantine equation P(x) =n! And a result of M.Overholt. Glasnik Matematicki, Vol 37(57), pp 269-273. Available at: https://web.math.pmf.unizy.hr/glasnik/37-2/37/21-04.pdf/ (Accessed on February 2, 2022)
[63] Mackinnon, Ronald Joseph (1970): Elementary approaches to Waring's Problem. Education Doctorate Thesis submitted to the Oklahoma University. Available at: https://shareok.org/bitstream/handle/11244/26635/Thesis-1970D-M158e.pdf?sequence=1&isAllowed=y/ (Accessed on February 6, 2022)
[64] Mahler, Kurt (1957): On the Fractional Parts of the Powers of a rational Number II. Mathematica 4, pp 122-124.
[65] Mason, R.C (1984): Diophantine equation over function fields. London Math Soc. Lectures notes ser.96, Cambridge.Univ.Press.
[66] Masser, D.W. (1985): Open problems, in proceedings of the Symposium on analytic number theory, London, Imperial college, edited by Chen, W.W.L.
[67] Matson, Robert D. (2017):Brocard's Problem 4^th solution search utilizing quadratic residues. Available at :https://web.archive.org/web/20181006100943/http://unsolvedproblems.org/S99.pdf/ (Accessed on January 29, 2022)
[68] Mehl, Serge: Grand théorème de Fermat, n=3, 4,5. Available at: https://serge.mehl.fr/anx/th_fermat_gd3.html/ (Accessed on: February 3, 2022)
[69] Mihailescu, P. (2004): Primary cyclotomic units and a proof of Catalan's conjecture. J. Reine Angew. Math, 572, pp 167-195.
[70] Mihailescu, P. (2005): Reflection, Bernoulli numbers and the proof of Catalan's conjecture. European Congress of Mathematics. Eur. Math. Soc. Zurich, pp 325-340.
[71] Mochizuki, Schinichi: Schinichi Mochizuki's papers. Available at: https://www.kurims.kyoto-u.ac.jp/ (Accessed on February 20, 2022)

[72] Mochizuki, Schinichi (2018): Report on discussions, held during the period March 15-20, 2018 concerning inter-universal Teichmüller theory. Available at: http://www.kurims.kyoto-u.ac.jp/~motizuki/Rpt2018.pdf/(Accessed on January 28, 2022)
[73] Mochizuki, Schinichi (2018): Comments on the manuscript by Scholze-Stix concerning Inter-Universal Teichmüller theory. Available at: http://www.kurims.kyoto-u.ac.jp/~motizuki/Cmt2018-05.pdf/(Accessed on January 25, 2022)
[74] Mochizuki, Shinichi (2018): Comments on the manuscript (2018-08 version) by Scholze-Stix concerning inter-universal Teichmüller theory. Available at: http://www.kurims.kyoto-u.ac.jp/~motizuki/Cmt2018-08.pdf/(Accessed on January 28, 2022)
[75] Mordell, Louis Joel (1922): On the rational solutions of the indeterminate equation of the third and fourth degrees. Proc. Cambridge. Philos. Soc. Vol 21, pp179-192.
[76] Murty, Ram- Wong, Siman (2002): The abc-conjecture and prime divisors of the Lucas and Lehmer sequences. Available at : http://mast.queensu.ca/~murty/murty-wong.pdf/ (Accessed on January 31, 2022)
[77] Nagell, T (1920) : Note sur l'équation indéterminée$y^q = \frac{x^n-1}{x-1}$, Norsk. Mat. Tidsskr., 2, pp75-78.
[78] Narkiewicz, W. (1986) : Classical problems in number theory, vol. 62 of Monografie Matematyczne [Mathematical Monographs], Pa´nstwowe Wydawnictwo Naukowe (PWN), Warsaw. See also : Narkiewicz, W (2000) : Classical problems in number theory. Graduate Texts in mathematics, vol 95, springer Verlag, New York.
[79] Nitaj, Abderrahmane : the abc-conjecture home page. Available at : http://www.math.unicaen.fr/~Nitaj/abc.html/ (Accessed on February 5, 2022)
[80] Nitaj, Abderrahmane(1996) : La conjecture abc. Enseign. Math. (2), 42, n° 1-2, pp 3-24.
[81] Oesterlé, J. (1988): Nouvelles approches du théorème de Fermat. Astérisque, Tome 161-162. See also Séminaire Bourbaki, expo n°694, pp 165-186. Available at: http://www.numdam.org/item?id=SB_1987-1988_30_165_0/(Accessed on January 27, 2022)
[82] Overholt, Marius (1993): The Diophantine equation n! +1=m^2, Bull. London Math. Soc, Vol 42, p 104. Available at: https://doi.org/10.1112/blms/25.2.104/(Accessed on January 25, 2022)
[83] Pillai, S.S. (1931): On the inequality 0<$a^x - b^y \leq n$. J. Indian. Math.Soc 19, pp 1-11.
[84] Pillai, S.S. (1936): On $a^x - b^y = c$. J.Indian Math Soc (N.S), pp 119-122, 215.
[85] Pillai, S.S. (1945): On The equation $2^x - 2^y = 2^X + 3^Y$. Bull Calcutta. Math.Soc 37, pp 18-20.
[86] Ramanujan, S. (1913): Question 469, J. Indian Math.Soc, Vol 5, pp 59.
[87] Ramanujan, S. (1962): Collected papers, Chelsea, New York, pp 327.
[88] Roth, K.F (1955): Rational approximation to algebraic numbers. Mathematica. 2, 1-20.
[89] Saradha, N. (2012): Applications of explicit abc-conjecture on two Diophantine equations. Acta. Arith. 151, n°4, pp 401- 419.
[90] Schmidt, Wolfgang M. (1996). Diophantine approximations and Diophantine equations. Lecture Notes in Mathematics. Vol. 1467 (2nd ed.). Springer-Verlag. pp. 205–206.
[91] Scholze, Peter-Stix, Jacob (2018): Why abc is still a conjecture. Available at: http://kurims.kyoto-u.ac.jp/~motizuki/SS2018-08.pdf/(Accessed on January 28, 2022) This is an adapted version of their May report Available at: http://kurims.kyoto-u.ac.jp/~motizuki/SS2018-05.pdf/ (Accessed on January 28, 2022)
[92] Sghiar, Mohamed (2018):La preuve de la conjecture ABC. IOSR Journal of mathematics. Vol 14, Issue 4, Ver. I, pp 22-26. Available at: www.iosrjournals.org/ and https://hal.archives-ouvertes.fr/hal-01830367/(Accessed on January 27, 2022)
[93] Sheppard, Joseph (2016): The abc-conjecture and its applications. A report submitted in partial fulfillment of the requirements for the degree Master of Science. Department of mathematics. College of arts and sciences. Kansas State University. Manhattan, Kansas. Approved by Pr. Christopher Pinner. Available at: http://core.ac.uk/download/pdf/77977806.pdf/ (Accessed on January 29, 2022)

[94] Shorey, T.N (1999) : Exponential Diophantine equations involving products of consecutive integers and related equations, Number Theory ed R.P. Bambah, V.C. Dumir and R.J. Hans-Gill, Hindustan Book Agency, pp 463-495.
[95] Shorey, T.N (2002) : Powers in arithmetic progression, In G. Wustholz (ed.) A panorama in Number theory, or the view from Baker's Garden, Cambridge Univ. Press, pp 325-336.
[96] Shorey, T.N (2002) : Powers in arithmetic progression II, Analytic Number Theory, RIMS Kokyuroku, Kyoto University.
[97] Shorey, T.N.-Saradha, N. (2005) : Contributions towards a conjecture of Erdos on perfect powers in arithmetic progressions, Compositio Math, 141, pp 541-560.
[98] Silverman, Joseph H. (1988): Wieferich criterion and the abc-conjecture. Journal of number theory. 30(2), pp 226-237. Available at: https://doi.org/10.1016%2F0022-314X%2888%2990019-4 Available at: https://zbmath.org/?format=complete&q=an:0654.10019/ (Accessed on January 27, 2022)
[99] Silverman, Joseph H. (2010): Lang's height conjecture and Szpiro's conjecture. New York Journal of Mathematics.16, pp 1-12. Available at: http://nyjm.albany.edu/2010/16-1p.pdf/ (Accessed on February 5, 2022)
[100] Stemmler, R. M (1990): The Ideal Waring theorem for Exponents 401 200000. Math. Comput, 55, 815-820.
[101] Stewart, C.L- Tijdeman, A (1986): On the Oesterlé-Masser conjecture. Monatshefte fur Mathematik, 102 (3), pp 251-257.
[102] Stewart, C.L-Yu, Kunrui (1991): On the abc-conjecture. Part I. Math. Ann, Vol 291, pp 225-230.
[103] Stewart, C.L-Yu, Kunrui (2001): on the abc-conjecture. Part II. Duke Math. Journal, Vol 108, number 1.
[104] Stewart, Cameron L. (2012): Exceptional units and cyclic results. Acta. Arithmetica 155 Issue 4, pp 407-418. Available at: http://eudml.org/doc/2788567/ (Accessed on January 31, 2022)
[105] Stothers, W (1981): Polynomial identities and Hauptmoduln. Quart. J. math. Oxford ser 2, volume32, pp 349-370.
[106] Suomalainen, Janne (2016): Waring's Problem. Master's Thesis submitted to the Helsinki University. Available at: https://helda.helsinki.fi/bitstream/handle/10138/166733/gradu.pdf/?sequence=3&isAllowed=y/ (Accessed on February 6, 2022)
[107] *Szpiro, L. (1981) : Séminaire sur les pinceaux des courbes de genre au moins deux, Astérisque,* ***86*** *(3), pp 44–78.*
[108] *Szpiro, L. (1987) : Présentation de la théorie d'Arakelov, Contemp. Math.* ***67,*** *pp 279–293.*
[109] Tijdeman, Robert (1976): On the equation of Catalan, Acta Arithmetica, **29** (2): pp 197–209.
[110] Vojta, Paul (1987): Diophantine approximation and value distribution theory. Lecture Notes in Math. Vol 1239, Springer-Verlag, Berlin.
[111] Waldschmidt, Michel: (2013): lecture on the abc-conjecture and some of its consequences; Abdussalam School of mathematical sciences, Lahore 6Th World conference on 21st Century Mathematics. Université Pierre Et Marie Curie- Paris. Available at: http://webusers.imj-prg.fr/~michel.waldscmidt/articles/pdf/abcLahoreProceedings;pdf/ (Accessed on January 28, 2022)
[112] Waldschmidt, Michel (2016): Conjecture abc: quelques consequences. Aims Senegal. M'Bour. African institute of mathematical sciences. Available at: https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/abcFrVI.pdf/ (Accessed on January 27, 2022)
[113] Waring, Eduard (1770-1782): Meditationes algebraicae M.Cambridge, England: pp204-205, 1770. Reprinted as, Meditationes algebraicae 3rd ed. Cambridge pp 349-350. 1782. Reprinted as: Meditationes Algebraicae: An English Translation of the work of Edward Waring. Providence, RI: Amer. Math.Soc., 1991.

[114] Wieferich, A. (1909): Zum lezten Fermat'schen Theorem. J. Reine Angew. Math, 136, pp 293-302.
[115] Wikipedia: the intermediate value theorem. Available at: https://en.wikipedia.org/wiki/Intermediate_value_theorem/(Accessed on February 2, 2022)
[116] Wikipedia: the Fundamental Arithmetical theorem. Available at: https://en.wikipedia.org/wiki/Fundamental_Arithmetical_theorem/(Accessed on February 1, 2022)
[117] Wikipedia: Bezout theorem. Available at: https://en.wikipedia.org/wiki/Bezout_theorem/(Accessed on February 1, 2022)
[118] Wikipedia: contrapositive proposition. Available at: https://en.wikipedia.org/wiki/contrapositive_proposition/(Accessed on February 3, 2022)
[119] Wikipedia: The l'Hôpital rule. Available at: https://en.wikipedia.org/wiki/L'Hopital_rule/(Accessed on February 3, 2022)
[120] Wikipedia: The ABC conjecture. Available at: https://en.wikipedia.org/wiki/ABC_conjecture/(Accessed on February 3, 2022)
[121] Wikipedia: Radical of an integer. Available at: https://en.wikipedia.org/wiki/Intermediate_value_theorem/(Accessed on February 3, 2022)
[122] Wikipedia: Stirling's approximation. Available at: https://en.wikipedia.org/wiki/Stirling_approximation/(Accessed on January 29, 2022)
[123] Wikipedia: Integer part. Available at: https://en.wikipedia.org/wiki/Integer_part/(Accessed on January 29, 2022)
[124] Wikipedia: Néron Model. Available at: https://en.wikipedia.org/wiki/Neron_model/(Accessed on January 29, 2022)
[125] Wikipedia: Division and divisor. Available at: https://en.wikipedia.org/wiki/Division_and_divisor/(Accessed on January 29, 2022)
[126] Wikipedia: Prime integer. Available at: https://en.wikipedia.org/wiki/Prime_Integer/(Accessed on January 29, 2022)
[127] Wikipedia: The greatest common divisor. Available at: https://en.wikipedia.org/wiki/grearest_common_divisor/(Accessed on January 29, 2022)
[128] Wikipedia: The lowest common multiple. Available at: https://en.wikipedia.org/wiki/lowest_common_multiple/(Accessed on January 29, 2022)
[129] Wikipedia: Circular functions. Available at: https://en.wikipedia.org/wiki/Circular_functions/(Accessed on January 29, 2022)
[130] Wikipedia: Bolzano-Weierstrass theorem. Available at: https://en.wikipedia.org/wiki/Bolzano_Weierstrass_theorem/(Accessed on January 29, 2022)
[131] Wikipedia: Hall's conjecture Available at: https://en.wikipedia.org/wiki/Hall_conjecture/(Accessed on January 29, 2022)
[132] Wiles, Andrew (1995): Modular elliptic curves and Fermat's last theorem. Annals of Mathematics. 142, pp 443-551. Available at: http://math.stanford.edu/~lekheng/flt/wiles.pdf (Accessed on January 28, 2022)
[133] Woods, Alan Robert (1981): *Some problems in logic and number theory, and their connections*, Doctorate Thesis Manchester University. Available at:https://archive.wikiwix.com/cache/index2.php?url=http%3A%2F%2Fschool.maths.uwa.edu.au%2F%7Ewoods%2Fthesis%2FWoodsPhDThesis.pdf%2Findex.html#federation=archive.wikiwix.com/(Accessed on January 29, 2022)