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CONFIRMATION OF THE FIROOZBAKHT CONJECTURE AND DEDUCTION OF THE ANDRICA, CRAMER AND SINHA CONJECTURES WITH SOME OTHER IMPORTANT CONSQUENCES

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ABSTRACT

I confirm, in the present short paper, the Firoozbakht conjecture, remained open since 1982. Recall that this conjecture is one of the Neil Sloane unsolved prime conjectures list. My Proof of the Firoozbakht conjecture is very simple and uses elementary tools of mathematics. Some important consequences, such as the Andrica (remained open since 1986), Cramér (remained open since 1936) and Sinha (remained open since 2018) conjectures are deduced from the Firoozbakht conjecture. The Deduction is based on the Taylor formula, the intermediate value theorem, the L'Hôpital rule, the Dursat theorem and the growth properties of some elementary functions.

KEYWORDS: Firoozbakht conjecture-Prime Number theorem-The intermediate value theorem-A decreasing function-An increasing function-Dusart theorem-Euclid Theorem-equivalent sequences-Andrica conjecture-Cramer conjecture-Sinha conjecture-Taylor formula with Lagrange rest 2010 Mathematics Subject Classification: 11 A xx (Elementary Number Theory).

INTRODUCTION

Definition1: We call Firoozbakht Conjecture the following assertion « if $(p_n)_{n\geq 1}$ denotes the infinite strictly increasing sequence of prime integers then the sequence of general term: $q_n = p_n^{\frac{1}{n}}$ is strictly decreasing i.e. $\forall n \geq 1\ q_{n+1} < q_n$

Definition 2: We call Andrica conjecture the following assertion "$\forall n \geq 1$: $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$"
Definition3: We call the strong Cramer conjecture the following assertion"$\limsup_{n\to+\infty} \frac{p_{n+1}-p_n}{(\ln(p_n))^2} = 1$"
Definition4: We call Sinha conjecture the following assertion " $\forall n > 4 : p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1$"
Some History: 1) **Firoozbakht conjecture:** The Firoozbakht conjecture was announced by the women Iranian Mathematician Farideh Firoozbakht (1962-2019) in 1982 [9], [17], [20], [21], [26]. Faredeh was a mathematical professor at the Isfahan University going from pharmacology specialty.

This conjecture is a part of the unsolved prime conjectures listed by Neil Sloane (classed as conjecture No 30). This list contains the Riemann Hypothesis proved by M. Ghanim in a paper published by the GJAETS on April 10, 2017 [11] and contains also the Goldbach conjecture and the twin prime conjecture proved by M.Ghanim in two papers published by the GJAETS on May 10, 2018 [13] and on July 10, 2018 [14] respectively.
By using a table of maximal gaps, Farideh Firoozbakht verified her conjecture up to 4.444×10^{12} [26]. Now with more extensive tables of maximal gaps, the conjecture has been verified for all primes below 4.10^{18}[18] And later below $2^{64} \approx 1.84\times10^{19}$ [26].

Firoozbakht wrote about herself: « I was born in 1962 in Isfahan, Iran. Since I was seventeen, because of my deep affection for the animals, I became a vegetarian.

After graduating from high-school, I went to the University of Isfahan to continue my studies in pharmacology which is one of the most favored disciplines in Iran. But, since I was very fond of mathematics, especially number theory, I changed my major to mathematics in the third year and I became graduated in 1987.

Afterwards, I went to Isfahan University of Technology to continue my postgraduate studies in mathematics.

Since 1992, I have been instructing mathematics at the Iranian universities and currently I am teaching at the University of Isfahan.

I have done research in number theory and I have been able to find beautiful and interesting relations. My favorite field is "The Gap between Prime Numbers."

Conjecture No. 30, came to mind in 1982, while I was studying the proof of the "Prime Number Theorem". I think that the conjecture is the most important conjecture related to prime numbers, it shows one of the most interesting and beautiful behaviors of the prime numbers and I believe working on this conjecture could guide us to important results in number theory ».[9]

2) Andrica conjecture : This conjecture was announced in 1986 (See [1]) by the Romanian mathematician (born in 1956) Dorin Andrica. The conjecture was numerically confirmed for any integer n such that: $1 \leq n \leq 4.10^{18}$.The Andrica conjecture is generalized by considering the equation : $p_{n+1}^x - p_n^x = 1$ where x can be any positive number. The maximal solution is $x_{max} = 1$ as it can be seen easily (occuring for n=1). The minimal solution is conjectured to be $x_{min} = 0.567148 \ldots$ occuring for $n = 30$. This last conjecture can be stated as :$p_{n+1}^x - p_n^x < 1$ for $x < x_{min} = 0.567148 \ldots$

3) Cramér Conjecture : In 1920, the Suidish mathematician Harald Cramér (1893-1985) proved (see [4]), under the Riemann hypothesis, the weak form $p_{n+1} - p_n = O\left(\sqrt{p_n}\ln(p_n)\right)$($O$ denoting the Bachmann-Landau symbol defined below). Currently the unconditional result is $p_{n+1} - p_n = O({p_n}^{0.535})$ due to R.Baker and G. Harman (See [2]) and $p_{n+1} - p_n = O({p_n}^{0.525})$ due to R.Baker, G. Harman and J. Pintz (See [3]). Cramer conjectured, using probabilistic methods, the strong form $\limsup_{n\to+\infty} \frac{p_{n+1}-p_n}{(\ln(p_n))^2} = 1$ in 1936 (See [5]).

4) Granville Conjecture : In 1995, the British Mathematician Andrew Granville (Born In 1962) has affined (see [15]) the initial Cramér conjecture by conjecturing that :$\limsup_{n\to+\infty} \frac{p_{n+1}-p_n}{(\ln(p_n))^{é}} = 2e^{-\gamma} = 1.1229 \ldots$ Where $\gamma = 0.5772 \ldots$denotes the Euler-Mascheroni Constant (M. Ghanim has resolved-in the affirmative- the problem of irrationality of γ remained open since 1734 by a paper published in the GJAETS in the May 10, 2017 issue [12]). This conjecture is false as it will be showed in the present paper.

5) Sinha conjecture : In 2018, the Indian Mathematician Sinha Niloptal Kanti has showed (See [19]), supposing the Firoozbakht conjecture true, what I call here the Sinha conjecture.

The Note: The present short note gives an elementary proof of the Firoozbakht conjecture based on simple tools of mathematics. Also some important consequences are deduced, such as the Andrica conjecture remained open since 1986, the Cramer conjecture remained open since 1936 and the Sinha conjecture remained open since 2018.

The organization of the paper: The paper is organized as follows. §1 is an introduction giving the necessary definition and some History. The §2 gives the ingredients of the proofs. The §3 gives the proof of the Firoozbakht conjecture. The §4 gives the proofs of some consequences of the conjecture. The § 5 gives some references for further reading.

INGREDIENTS OF THE PROOFS

Definition 5: A positive integer p is called to be prime if its set of divisors is $D(p) = \{1, p\}$
We denote by $\mathbb{P}$ the set $\{p \in \mathbb{N}, p \text{ is prime}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ of prime integers and for any real number $x \geq 2$ by $\mathbb{P}(x)$ the set $\{p \in \mathbb{P}, p \leq x\}$ and by $\pi(x) = card(\mathbb{P}(x))$ the cardinal of $\mathbb{P}(x)$ i.e. the number of its elements.

Proposition1: (Euclid) [8] the set $\mathbb{P}$ is an infinite strictly increasing sequence $(p_n)_{n\geq1}$

Proposition2 : $\forall n \geq 1$ we have: $\pi(p_n) = n$

Proposition3 : (Prime number theorem (PNT)) (Hadamard [16]-De La valée Poussin [6]), we have :

$$\lim_{t\to+\infty} \frac{\pi(t)\ln(t)}{t} = 1$$

Remark: The PNT was proved independently in 1896 by the French mathematician Jacques Salomon Hadamard (1865-1965) and the Belgian mathematician Charles Jean De La Valée Poussin (1866-1962)

Proposition4 : (Dusart theorem (See The assertion (5) of theorem 1.10 in [7])) We have :
(i)$\forall x \geq 5393 \quad \frac{x}{\pi(x)} \leq \ln(x) - 1$
(ii)$\forall x \geq 599 \quad \frac{x}{\ln(x)}(1 + \frac{1}{\ln(x)}) \leq \pi(x)$
(iii)$\forall x \geq 355991 \quad \pi(x) \leq \frac{x}{\ln(x)}(1 + \frac{1}{\ln(x)} + \frac{2.51}{(\ln(x))^2})$

Proposition5: (continuity) [33]: (i) $U \subset \mathbb{R}$ is open if: $\forall a \in U \exists \delta > 0\ ]a - \delta, a + \delta[\subset U$
(ii) $\lim_{t\to a} f(t) = l \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0\ \forall t \in U\ \ a - \delta < t < a + \delta \Rightarrow l - \epsilon < f(t) < l + \epsilon$
(iii)$f: U \to \mathbb{R}$ is continuous in a point a $\Leftrightarrow \lim_{t\to a} f(t) = f(a)$
(iv)$f: U \to \mathbb{R}$ is continuous$\Leftrightarrow$ $\forall O$ an open subset of $\mathbb{R}$ $f^{-1}(O) = \{t \in U, f(t) \in O\}$ is an open subset of U.

Proposition 6: (connected spaces) [35] (i) an open subset U of a topological space is connected if:
$$\forall A, B \text{ open subsets of } U \begin{cases} AUB = U \\ A \cap B = \emptyset \end{cases} \Rightarrow A = U \text{ or } B = U$$
(ii)The connected parts of $\mathbb{R}$ are the intervals.

Proposition7: (the intermediate value theorem) [27] let $f: [a, b](a < b) \to \mathbb{R}$ a continuous function. Then:
$$f(a)f(b) < 0 \Rightarrow \exists c \in]a, b[\ f(c) = 0$$
If f is increasing or decreasing in $]a, b[$: c is the sole zero of f in $]a, b[$.

Proposition8: (infimum and supremum) [28] (i) we denote by inf (A) the infimum of the set A i.e. the greatest lower bound of A and by sup (A) the supremum of A i.e. the smallest upper bound.
(ii) Any non empty part A of $\mathbb{N}$ has an inf (A)$\in$A
(iii) l is an adherent point of the sequence $(x_n)_n \Leftrightarrow \exists \varphi: \mathbb{N} \to \mathbb{N}$ a bijection such that: $\lim_{n\to+\infty} x_{\varphi(n)} = l$
(iv)We define: $\liminf_{n\to+\infty} x_n = \sup_{m\geq m_0} \inf_{n\geq m} x_n \in [-\infty, +\infty]$
(v) $\liminf_{n\to+\infty} x_n$ is the smallest adherent point.
(vi)We define: $\limsup_{n\to+\infty} x_n = \inf_{m\geq m_0} \sup_{n\geq m} x_n \in [-\infty, +\infty]$
(vii)$\mathrm{limsupf}_{n\to+\infty} x_n$ is the greatest adherent point
(viii)We have: $\forall n \geq m_0\ x_n < a \Rightarrow \limsup_{n\to+\infty} x_n \leq a$
(ix)We have: $\forall n \geq m_0\ x_n > b \Rightarrow \liminf_{n\to+\infty} x_n \geq b$

Proposition9: (The Greatest common divisor) [23] (i) Recall that: gcd (a, b) denotes the positive greatest common divisor of the integers a, b
(ii) Example: $\forall n \geq 1\ \gcd(p_n, p_{n+1}) = 1$
(iii)$\gcd(a, b) = 1 \Leftrightarrow \forall m, n \geq 1\ \gcd(a^n, b^m) = 1$
(iv)(The Bezout theorem): $\gcd(a, b) = 1 \Leftrightarrow \exists u, v \in \mathbb{Z}$ such that: $ua + vb = 1$

Proposition10: (The Arithmetical Fundamental Theorem) [25] we have:
$$\forall n \text{ integer } \geq 2 \exists (\alpha_k)_{k\geq 1} \text{a sequence of natural integers such that:} \begin{cases} \text{the set } \{k, \alpha_k \neq 0\} \text{is finite} \\ n = \prod_{k=1}^{+\infty} p_k^{\alpha_k} \end{cases}$$

Proposition11: (Increasing and decreasing functions) [31], Recall that:
(i) a function f is derivable in a point s if $\lim_{t\to s} \frac{f(t)-f(s)}{t-s} = f'(s) \in \mathbb{R}$
(ii) A function f is strictly decreasing on $]a, b[\Leftrightarrow \forall x, y \in]a, b[: x < y \Rightarrow f(y) < f(x)$
(iii) A function f is strictly increasing on $]a, b[\Leftrightarrow \forall x, y \in]a, b[: x < y \Rightarrow f(x) < f(y)$
(iv) If f is a derivable function of derivativef', then:
$$f \text{ Strictly decreasing on}]a, b[\Leftrightarrow \forall t \in]a, b[\ f'(t) < 0$$
(v) If f is a derivable function of derivativef', then:
$$f \text{Strictly increasing on}]a, b[\Leftrightarrow \forall t \in]a, b[\ f'(t) > 0$$

Proposition12: (The Taylor Formula with Lagrange rest) [29] suppose that the function f has a continuous derivative f' in the interval $[a,b]$ and that the second derivative f'' exists in the interval$]a,b[$, then $\forall x \in [a,b] \exists c \in]a,x[\ f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(c)}{2!}(x-a)^2$

Proposition13: (the l'Hôpital rule) [30] if $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$ with A, B are both null or are both infinite, we say that $\lim_{x\to c}\frac{f(x)}{g(x)}$ is an indefinite form IF $\frac{0}{0}$ or $\frac{\infty}{\infty}$

(i) If f, g are derivable on the interval$]\ a, b[$except perhaps in a point $c \in]a,b[$ and $\lim_{x\to c}\frac{f(x)}{g(x)}$ is IF$\frac{0}{0}$ with : $g'(x) \neq 0\ for\ x \neq c$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

(ii) The result of (i) is true if $\lim_{x\to c}\frac{f(x)}{g(x)}$ is IF$\frac{\infty}{\infty}$

(iii) If f', g' satisfy the same conditions, the process is repeated.

(iv) The result is extended to the cases : $x \to \infty, x \to a^+, x \to b^-$

Proposition14: (equivalent sequences) [32] we have:
(ii) $\lim_{n\to+\infty} x_n = l \Leftrightarrow \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N\ \ l-\epsilon < x_n < l+\epsilon$
(ii) Two sequences x_n, y_n are equivalent if $\lim_{n\to+\infty}\frac{x_n}{y_n} = 1$
(iii)If x_n, y_n are equivalent, then $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n$

Proposition15: (The Bolzano-Weierstrass Theorem) [34] any bounded sequence $(x_n)_n \subset]a,b[$ has a convergent subsequence denoted, $(x_{\varphi(n)})_n$, with:$\varphi(n) > n\ (\forall n)$ and $\lim_{n\to+\infty} x_{\varphi(n)} \in [a,b]$.

Definition6: (The Bachman-Landau symbol) [36] we have:
$g(x) = O(f(x))$ when $x \to +\infty \Leftrightarrow \exists N \exists c > 0 \forall x \geq N\ |g(x)| < c|f(x)|$

Proposition16: (the twin prime conjecture) [9] [14] we have:
$\exists \varphi; \mathbb{N} \to \mathbb{N}$ a bijection such that $\forall n \in \mathbb{N}\ p_{\varphi(n)+1} - p_{\varphi(n)} = 2$

Proposition17: (E. Westzynthius theorem) [22] we have $\limsup_{n\to+\infty}\frac{p_{n+1}-p_n}{\ln(p_n)} = +\infty$

Proposition19: (The squeeze theorem (or pinching theorem or sandwich rule) [37] Let E a topological space. Let A a subset of E. Let a an adherent point of A (i.e. $\forall O$ an open subset of E containing a we have $O \cap A \neq \emptyset$). Letf, g, h:$A \to [-\infty, +\infty]$, three functions such that $f \leq g \leq h$ on A. if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \in [-\infty, +\infty]$, then: g converges in a and $\lim_{x\to a} g(x) = L$

Proposition20: (i) The sinha conjecture (see definition 4 above) can be confirmed numerically for $5 \leq n \leq 700$
(ii) The Andrica conjecture (see definition2 above) was confirmed numerically for $1 \leq n \leq 4.10^{18}$ (See [38])

PROOF OF THE FIROOZBAKHT CONJECTURE

Theorem: (Firoozbakht conjecture) the sequence of general term $q_n = p_n^{\frac{1}{n}}$ is strictly decreasing. That is $\forall n \geq 1$ we have: $q_{n+1} = p_{n+1}^{\frac{1}{n+1}} < q_n = p_n^{\frac{1}{n}}$
Proof: (of the theorem)
The Proof of the theorem will be deduced from the claims below.

Claim1: We have: $\forall n \geq 1\ \ p_n^{\frac{1}{n}} \neq p_{n+1}^{\frac{1}{n+1}}$
Proof: (of claim1)

*Suppose contrarily that: $\exists n \geq 1$ such that: $p_n^{\frac{1}{n}} = p_{n+1}^{\frac{1}{n+1}}$
*That is: $p_n^{n+1} = p_{n+1}^n$
*By the assertions (ii) and (iii) of proposition9, we have : $\gcd(p_n, p_{n+1}) = 1 \Rightarrow \gcd(p_n^{n+1}, p_{n+1}^n)$=1

*So, by the Bezout theorem (the assertion (iv) of proposition9), $\exists u, v \in \mathbb{Z}$ such that: $up_n^{n+1} + vp_{n+1}^n = 1 = up_n^{n+1} + vp_n^{n+1} = (u+v)p_n^{n+1}$
*So: $p_n = 1$
*This being impossible, because $\forall n \geq 1 p_n \geq 2$, the result follows
Definition7: for $\epsilon \in]0,1[, n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$, define the sequence of continuous functions:

$$\epsilon \to \alpha_{n,m}(\epsilon) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$$

Claim2: $\forall \epsilon \in]0,1[\; \forall n \in \mathbb{N}^* \forall m \in \mathbb{N}^*$ we have: $-\epsilon - \frac{1}{m} < \alpha_{n,m}(\epsilon) < \epsilon - \frac{1}{m}$
Proof: (of claim2)

*We have: $\alpha_{n,m}(\epsilon) - \left(\epsilon - \frac{1}{m}\right) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} - \left(\epsilon - \frac{1}{m}\right) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}} - \left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}\right)}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$

$= \frac{-\left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}} - \left(\epsilon - \frac{1}{m}\right)\left(+p_n^{\frac{1}{n}}\right)}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} = -2\epsilon \frac{p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} < 0$

*We have: $\alpha_{n,m}(\epsilon) + \left(\epsilon + \frac{1}{m}\right) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} + \left(\epsilon + \frac{1}{m}\right) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}} + \left(\epsilon + \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}\right)}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$

$= \frac{\left(\frac{1}{m} - \epsilon\right)p_{n+1}^{\frac{1}{n+1}} + \left(\epsilon + \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right)}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} = 2\epsilon \frac{p_{n+1}^{\frac{1}{n+1}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} > 0$

*The result follows.

Claim3: (i)$\forall \epsilon \in]0,1[\; \forall n \in \mathbb{N}^* \forall m \in \mathbb{N}^*$, we have: $\alpha_{n,m}(\epsilon) = -\frac{1}{m} + \epsilon \frac{p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$

(ii) $\forall \epsilon \in]0,1[\; \forall n \in \mathbb{N}^* \; \lim_{m \to +\infty} \alpha_{n,m}(\epsilon) = \epsilon \frac{p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$

Proof: (of claim3)

(i) By definition, we have: $\alpha_{n,m}(\epsilon) = \frac{\left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - \left(\frac{1}{m} + \epsilon\right)p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} = \frac{-\frac{1}{m}\left(p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}\right) + \epsilon\left(p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}\right)}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} = -\frac{1}{m} + \epsilon \frac{p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}}$

(ii)The assertion Follows immediately form the assertion (i) of claim3.
Claim4: $\forall n \in \mathbb{N}^* \exists m_n \in \mathbb{N}^*$such that $\forall \epsilon \in]\frac{1}{n+1}, 1[$ we have: $\frac{1}{m_n} < -\alpha_{n,m_n}(\epsilon)$ or $\alpha_{n,m_n}(\epsilon) > 0$
Proof: (of claim4)
*Suppose contrarily that: $\exists n \in \mathbb{N}^* \forall m \in \mathbb{N}^* \exists \epsilon_m \in]\frac{1}{n+1}, 1[\frac{1}{m} \geq -\alpha_{n,m}(\epsilon_m) \geq 0$
*By the Bolzano-Weierstrass :($\epsilon_m)_m \subset]\frac{1}{n}, 1[$ has a convergent subsequence $(\epsilon_{\varphi(m)})_m$, such that $\varphi(m) > m \; \forall m \geq 1$ and $\epsilon = \lim_{m \to +\infty} \epsilon_{\varphi(m)} \in [\frac{1}{n+1}, 1]$,
*We have: $\forall m \in \mathbb{N}^*: \frac{1}{\varphi(m)} \geq -\alpha_{n,\varphi(m)}\left(\epsilon_{\varphi(m)}\right) \geq 0$

*So, tending $m \to +\infty$: we have: $0 = \lim_{m \to +\infty} \frac{1}{\varphi(m)} \geq -\lim_{m \to +\infty} \alpha_{n,\varphi(m)}\left(\epsilon_{\varphi(m)}\right) = -\epsilon \frac{p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} \geq 0$

*That is $-\epsilon \frac{p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} = 0$ i.e. $\epsilon = 0$ *or* $p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}} = 0$

*But "$\epsilon = 0$" is impossible because: $\epsilon \in [\frac{1}{n+1}, 1] \Rightarrow \epsilon \geq \frac{1}{n+1} > 0$ and "$p_{n+1}^{\frac{1}{n+1}} - p_n^{\frac{1}{n}} = 0$" is impossible by claim1.
(assuring that: $\forall n \geq 1 \; p_n^{\frac{1}{n}} \neq p_{n+1}^{\frac{1}{n+1}}$)

*So, our starting absurd hypothesis "$\exists n \in \mathbb{N}^* \; \forall m \in \mathbb{N}^* \exists \epsilon_m \in]\frac{1}{n+1}, 1[\, 0 \leq -\alpha_{n,m}(\epsilon) \leq \frac{1}{m}$" is false, so its negation: "$\forall n \in \mathbb{N}^* \exists m_n \in \mathbb{N}^* \forall \epsilon \in]\frac{1}{n+1}, 1[: -\alpha_{n,m_n}(\epsilon) > \frac{1}{m_n}$ or $\alpha_{n,m_n}(\epsilon) > 0$" is true.

<u>Claim5:</u> $\forall n \in \mathbb{N}^* \exists m_n \in \mathbb{N}^*$ such tha $\forall \epsilon \in]\frac{1}{n+1}, 1[$ we have: $-\alpha_{n,m_n}(\epsilon) > \frac{1}{m_n}$

<u>Proof:</u> (of claim5)

*Consider, for any $n \in \mathbb{N}^*$, the subsets:

$$A_n = \{\epsilon \in]\frac{1}{n+1}, 1[: -\alpha_{n,m_n}(\epsilon) > \frac{1}{m_n}\} \text{ and } B_n = \{\epsilon \in]\frac{1}{n+1}, 1[: \alpha_{n,m_n}(\epsilon) > 0\}$$

*The function $\epsilon \to \alpha_{n,m_n}(\epsilon)$ being continuous: A_n and B_n are open subsets of$]\frac{1}{n+1}, 1[$

*By claim4: $]\frac{1}{n+1}, 1[= A_n \cup B_n$

*We have: $A_n \cap B_n = \emptyset$, because: $0 > -\alpha_{n,m_n}(\epsilon) > \frac{1}{m_n} > 0$ is impossible

*So, $]\frac{1}{n+1}, 1[$ being connected, we have: $\forall n \in \mathbb{N}^* \; A_n =]\frac{1}{n+1}, 1[$ *or* $B_n =]\frac{1}{n+1}, 1[$

<u>First case:</u> $B_n =]\frac{1}{n+1}, 1[$

*So: $\forall n \in \mathbb{N}^* \exists m_n \in \mathbb{N}^* \forall \epsilon \in]\frac{1}{n+1}, 1[\, \alpha_{n,m_n}(\epsilon) > 0$

*In particular, because:$p_1 = 2$ *and* $p_2 = 3$, we have:

$$\text{For } n = 1, \exists m \in \mathbb{N}^* \forall \epsilon \in]\tfrac{1}{2}, 1[\, 0 < \alpha_{2,m}(\epsilon) = -\frac{1}{m} + \epsilon \frac{p_2^{\frac{1}{2}} - p_1}{p_2^{\frac{1}{2}} + p_1} = -\frac{1}{m} + \epsilon \frac{3^{\frac{1}{2}} - 2}{3^{\frac{1}{2}} + 2} = -\frac{1}{m} - 0.071.\epsilon < 0$$

*This being impossible, this case cannot occur. So the below second case is true:

<u>Second case:</u> $A_n =]\frac{1}{n+1}, 1[$

*That is: $\forall n \in \mathbb{N}^* \exists m_n \in \mathbb{N}^* \forall \epsilon \in]\frac{1}{n+1}, 1[$ we have: $-\alpha_{n,m_n}(\epsilon) > \frac{1}{m_n}$

*So, claim 5 is proved.

<u>Claim6:</u> (i) $\forall n, m \in \mathbb{N}^* \forall \epsilon \in]0,1[$, we have: $0 < \epsilon - \frac{1}{m} - \alpha_{n,m}(\epsilon)$

(ii) $\forall n, m \in \mathbb{N}^* \forall \epsilon \in]0,1[\; \frac{p_{n+1}^{\frac{1}{n+1}}}{p_n^{\frac{1}{n}}} = \frac{\epsilon + \frac{1}{m} + \alpha_{n,m}(\epsilon)}{\epsilon - \frac{1}{m} - \alpha_{n,m}(\epsilon)}$

<u>Proof:</u> (of claim6)

(i)The result follows immediately from the right inequality of claim2.

(ii)We have:

$$\alpha_{n,m}(\epsilon) = \frac{(\epsilon - \frac{1}{m})\left(p_{n+1}^{\frac{1}{n+1}}\right) - (\frac{1}{m} + \epsilon) p_n^{\frac{1}{n}}}{p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}} \Leftrightarrow \alpha_{n,m}(\epsilon)\left(p_{n+1}^{\frac{1}{n+1}} + p_n^{\frac{1}{n}}\right) = \left(\epsilon - \frac{1}{m}\right)\left(p_{n+1}^{\frac{1}{n+1}}\right) - (\frac{1}{m} + \epsilon) p_n^{\frac{1}{n}}$$

$$\Leftrightarrow p_n^{\frac{1}{n}}\left(\alpha_{n,m}(\epsilon) + \frac{1}{m} + \epsilon\right) = p_{n+1}^{\frac{1}{n+1}}\left(\epsilon - \frac{1}{m} - \alpha_{n,m}(\epsilon)\right) \Leftrightarrow \frac{p_{n+1}^{\frac{1}{n+1}}}{p_n^{\frac{1}{n}}} = \frac{\epsilon + \frac{1}{m} + \alpha_{n,m}(\epsilon)}{\epsilon - \frac{1}{m} - \alpha_{n,m}(\epsilon)}$$

RETURN TO THE PROOF OF THE THEOREM

*$\forall n \in \mathbb{N}^*$, by claim6, written for $m = m_n$ and $\epsilon \in]\frac{1}{n+1}, 1[$ given by claim 5, we have: $\frac{p_{n+1}^{\frac{1}{n+1}}}{p_n^{\frac{1}{n}}} = \frac{\epsilon + \frac{1}{m_n} + \alpha_{n,m_n}(\epsilon)}{\epsilon - \frac{1}{m_n} - \alpha_{n,m}(\epsilon)}$

*But, by claim5 and the assertion (i) of claim6, we have successively: $\forall n \in \mathbb{N}^*$

$$\begin{cases} 0 < \epsilon - \frac{1}{m_n} - \alpha_{n,m_n}(\epsilon) \\ \frac{1}{m_n} < -\alpha_{n,m}(\epsilon) \end{cases} \Rightarrow \frac{2}{m_n} + \epsilon < \epsilon - 2\alpha_{n,m_n}(\epsilon) \Rightarrow \frac{1}{m_n} + \epsilon + \alpha_{n,m_n}(\epsilon) < \epsilon - \frac{1}{m_n} - \alpha_{n,m_n}(\epsilon)$$

$$\Rightarrow \frac{\frac{1}{m_n} + \epsilon + \alpha_{n,m_n}(\epsilon)}{\epsilon - \frac{1}{m_n} - \alpha_{n,m_n}(\epsilon)} < 1 \Rightarrow \frac{p_{n+1}^{\frac{1}{n+1}}}{p_n^{\frac{1}{n}}} = \frac{\epsilon + \frac{1}{m_n} + \alpha_{n,m}(\epsilon)}{\epsilon - \frac{1}{m_n} - \alpha_{n,m}(\epsilon)} < 1 \Rightarrow \forall n \in \mathbb{N}^* p_{n+1}^{\frac{1}{n+1}} < p_n^{\frac{1}{n}}$$

*This finishes the proof of the theorem.

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CONSEQUENCES

1. PROOF OF SOME USEFULL LIMITS AND INEQUALITIES

Corollary1: We have (i) $\lim_{n\to+\infty} \frac{\sqrt{p_n}}{n} = 0$ (ii) $\lim_{n\to+\infty} \frac{n}{p_n} = 0$

(iii) $\lim_{n\to+\infty} \frac{\ln(p_n)}{n} = 0$

Proof: (of corollary1)

(i)*By the assertion (i) of the Dusart theorem, we have: $\frac{p_n}{n} = \frac{\sqrt{p_n}\sqrt{p_n}}{n} \leq \ln(p_n)$ for $p_n \geq 599$

*So: $0 < \frac{\sqrt{p_n}}{n} \leq \frac{\ln(p_n)}{\sqrt{p_n}}$

*But by the L'Hôpital rule: $\lim_{t\to+\infty} \frac{\ln(t)}{\sqrt{t}} = FI\frac{+\infty}{+\infty} = \lim_{t\to+\infty} \frac{(\ln(t))'}{(\sqrt{t})'} = \lim_{t\to+\infty} \frac{\frac{1}{t}}{\frac{1}{2\sqrt{t}}} = \lim_{t\to+\infty} \frac{2}{\sqrt{t}} = 0$

*So: $\lim_{n+\to\infty} \frac{\ln(p_n)}{\sqrt{p_n}} = 0$

*The result follows by the squeeze theorem.

(ii)*By the assertion (iii) of the Dusart theorem, we have:

$$0 < \frac{n}{p_n} \leq \frac{1}{\ln(p_n)}\left(1 + \frac{1}{\ln(p_n)} + \frac{2.51}{(\ln(p_n))^2}\right)\forall p_n \geq 355991$$

*The result follows by the squeeze theorem.

(iii)The result will be deuced from the claims below.

Claim7: We have: $\lim_{n\to+\infty} \frac{\frac{(\ln(p_n))^2}{p_n}}{\frac{\ln(p_n)}{n}} = 1$

Proof: (of claim7)

By combination of proposition2 and proposition3, we have:

$$\lim_{n\to+\infty} \frac{\frac{(\ln(p_n))^2}{p_n}}{\frac{\ln(p_n)}{n}} = \lim_{n\to+\infty} \frac{n\ln(p_n)}{p_n} = \lim_{n\to+\infty} \frac{\pi(p_n)\ln(p_n)}{p_n} = 1$$

Claim8 : We have : (i) $\lim_{t\to+\infty} \frac{\ln(t)}{\sqrt{t}} = 0$

(ii) $\lim_{n\to+\infty} \frac{(\ln(p_n))^2}{p_n} = 0$

Proof : (of claim8)

(i)By the L'Hôpital rule $\lim_{t\to+\infty} \frac{\ln(t)}{t^{\frac{1}{2}}} = FI\frac{\infty}{\infty} = \lim_{t\to+\infty} \frac{(\ln(t))'}{(t^{\frac{1}{2}})'} = \lim_{t\to+\infty} \frac{\frac{1}{t}}{\frac{1}{2}t^{-\frac{1}{2}}} = \lim_{t\to+\infty} \frac{2}{\sqrt{t}} = 0$

(ii)So: $\lim_{n\to+\infty} \frac{(\ln(p_n))^2}{p_n} = (\lim_{n\to+\infty} \frac{\ln(p_n)}{p_n^{\frac{1}{2}}})^2 = (\lim_{t\to+\infty} \frac{\ln(t)}{t^{\frac{1}{2}}})^2 = 0$

Claim9: We have:

(i) $\lim_{n\to+\infty} \frac{\ln(p_n)}{n} = 0$

(ii) $\lim_{n\to+\infty} p_n^{\frac{1}{n}} = 1$

Proof: (of claim9)

(i)The result follows by combination of proposition 14 (assuring that two equivalent sequences have the same limit), claim7 (assuring that the sequence of general term $\frac{\ln(p_n)}{n}$ is equivalent to the sequence $\frac{(\ln(p_n))^2}{p_n}$) and the assertion (ii) of claim8 (assuring that : $\lim_{n\to+\infty} \frac{(\ln(p_n))^2}{p_n} = 0$)

(ii) By the assertion (i) of claim9, we have:

* $\lim_{n\to+\infty} p_n^{\frac{1}{n}} = \lim_{n\to+\infty} e^{\frac{\ln(p_n)}{n}} = e^{\lim_{n+\to\infty} \frac{\ln(p_n)}{n}} = e^0 = 1$

Corollary2: $\lim_{n\to+\infty} \frac{\ln(p_{n+1})}{\ln(p_n)} = 1$

Proof: (of corollary 2)

*By the Theorem, we have: $\forall n \geq 1 \quad 1 \leq \frac{\ln(p_{n+1})}{\ln(p_n)} \leq \frac{n+1}{n}$

*So: $\lim_{n\to+\infty} 1 = 1 \leq \lim_{n\to+\infty} \frac{\ln(p_{n+1})}{\ln(p_n)} \leq \lim_{n\to+\infty} \frac{n+1}{n} = 1 \Rightarrow \lim_{n\to+\infty} \frac{\ln(p_{n+1})}{\ln(p_n)} = 1$

Corollary 3: We have: $\forall n \geq 1\ \ p_n \leq 2^n$

Proof: (of corollary 3)

*By multiplication, member to member, of the relations- deduced from the theorem- in the below system, we have successively: $\forall n \geq 1$

$$\begin{cases} \frac{\ln(p_2)}{\ln(p_1)} \leq \frac{2}{1} \\ \frac{\ln(p_3)}{\ln(p_2)} \leq \frac{3}{2} \\ \dots \\ \frac{\ln(p_{n-1})}{\ln(p_{n-2})} \leq \frac{n-1}{n-2} \\ \frac{\ln(p_n)}{\ln(p_{n-1})} \leq \frac{n}{n-1} \end{cases} \Rightarrow \frac{\ln(p_n)}{\ln(p_{n-1})}\frac{\ln(p_{n-1})}{\ln(p_{n-2})}\dots\frac{\ln(p_3)}{\ln(p_2)}\frac{\ln(p_2)}{\ln(p_1)} = \frac{\ln(p_n)}{\ln(p_1)} = \frac{\ln(p_n)}{\ln(2)} \leq \frac{n}{n-1}\frac{n-1}{n-2}\dots\frac{3}{2}\frac{2}{1} = n$$

$\Rightarrow \ln(p_n) \leq \ln(2^n) \Rightarrow p_n \leq 2^n$

Corollary 4: $\forall n \geq 109\ \ n < p_n < n^2\ln(2) < n^2$

Proof: (of corollary4)

*By corollary 3, we have: $\frac{p_n}{n\ln(p_n)} \geq \frac{p_n}{n\ln(2^n)} = \frac{p_n}{n^2\ln(2)}$

*But, by the Dusart theorem (See the assertion (ii) of proposition 4), we have: $\forall p_n > 599$: $\frac{p_n}{n\ln(p_n)} < 1$

*So: $\forall n \geq 109\ \ \frac{p_n}{n^2\ln(2)} < 1$

*The result follows.

Remark: (i) Show by recurrence that:$\forall n \geq 1\ n < p_n$

*We have: $n = 1 < p_1 = 2$

*Suppose that: $n < p_n$ and show that: $n + 1 < p_{n+1}$

*By the Euclid theorem (See proposition 1), we have:

$$\begin{cases} p_n < p_{n+1} \\ n < p_n \end{cases} \Rightarrow \begin{cases} p_n + 1 \leq p_{n+1} \\ n + 1 < 1 + p_n \end{cases} \Rightarrow n + 1 < 1 + p_n \leq p_{n+1} \Rightarrow n + 1 < p_{n+1}$$

(ii)We have: 599 is prime with $\pi(599) = 109$

Corollary 5: we have:

$$\forall n \text{ integer} \geq 2 \exists (\alpha_k)_{k\geq 1} \text{ a sequence of integers such that:} \begin{cases} \text{the set } \{k \geq 1, \alpha_k \neq 0\} \text{is finite} \\ n \leq 2^{\sum_{k=1}^{+\infty} k\alpha_k} \end{cases}$$

Proof: (of corollary 5)

*By the arithmetical fundamental theorem: $\forall n \geq 2 \exists (\alpha_k)_{k\geq 1}$such that:$\begin{cases} \text{the set } \{k, \alpha_k \neq 0\} \text{is finite} \\ n = \prod_{k=1}^{+\infty} p_k^{\alpha_k} \end{cases}$

*So, by corollary 3: $n = \prod_{k=1}^{+\infty} p_k^{\alpha_k} \leq \prod_{k=1}^{+\infty} 2^{k\alpha_k} = 2^{\sum_{k=1}^{+\infty} k\alpha_k}$

*So: $\forall n$ integer $\geq 2\exists (\alpha_k)_{k\geq 1}$ a sequence of integers such that:$\begin{cases} \text{the set } \{k \geq 1, \alpha_k \neq 0\} \text{is finite} \\ n \leq 2^{\sum_{k=1}^{+\infty} k\alpha_k} \end{cases}$

Corollary6 $\forall n \geq 1$ we have: $(\frac{\ln(p_{n+1})}{\ln(p_n)})^n < e$

Proof: (of corollary 6)

*By the theorem, we have:

$$\forall n \geq 1 \frac{\ln(p_{n+1})}{\ln(p_n)} < 1 + \frac{1}{n} \Rightarrow \forall n \geq 1 (\frac{\ln(p_{n+1})}{\ln(p_n)})^n < (1 + \frac{1}{n})^n$$

***Claim10:** We have: $\lim_{n\to+\infty} (1 + \frac{1}{n})^n = e$

Proof: (of claim10)

*By the L'Hôpital rule $\lim_{t\to 0} \frac{\ln(1+t)}{t} = FI\frac{0}{0} \Rightarrow \lim_{t\to 0} \frac{\ln(1+t)}{t} = \lim_{t\to 0} \frac{(\ln(1+t))'}{t'} = \lim_{t\to 0} \frac{\frac{1}{1+t}}{1} = 1$

*Letting $t = \frac{1}{n} \to 0$ for $n \to +\infty$, we have: $(1 + \frac{1}{n})^n = e^{n\ln(1+\frac{1}{n})} = e^{\frac{\ln(1+t)}{t}}$

*So: $\lim_{n\to+\infty} (1 + \frac{1}{n})^n = \lim_{t\to 0} e^{\frac{\ln(1+t)}{t}} = e^{\lim_{t\to 0}\frac{\ln(1+t)}{t}} = e^1 = e$

***Claim11:** We have: $\frac{1}{t+1} < \ln(1 + \frac{1}{t})\ \forall t > 0$

Proof: (of claim11)

*Consider the derivable function: $h(t) = \ln\left(1 + \frac{1}{t}\right) - \frac{1}{1+t}$

*We have: $\lim_{t\to+\infty} h(t) = \lim_{t\to+\infty} \ln\left(1+\frac{1}{t}\right) - \lim_{t\to+\infty} \frac{1}{1+t} = \ln(1) - 0 = 0$

*We have: $h'(t) = \frac{1}{(1+t)^2} - \frac{1}{t(t+1)} = \frac{t-(1+t)}{t(1+t)^2} = -\frac{1}{t(1+t)^2} < 0\ \forall t > 0$

*So: $\forall t > 0\ \ h(t) = \ln\left(1+\frac{1}{t}\right) - \frac{1}{1+t} > \lim_{t\to+\infty} h(t) = 0$

***Claim12** (i) the function $u(t) = t\ln\left(1+\frac{1}{t}\right)$ is strictly increasing for $t > 0$.

(ii)The sequence of general term: $u_n = \left(1+\frac{1}{n}\right)^n$ is strictly increasing.

(iii) $e = sup_{n\geq 1}\left(1+\frac{1}{n}\right)^n$

Proof: (of claim12)

(i)By claim10, we have: $u'(t) = \ln\left(1+\frac{1}{t}\right) - \frac{1}{1+t} > 0$, so the result follows.

(ii)The result follows immediately from claim12 (i).

(iii) The result follows by combination of claim10 and the assertion (ii) of claim12.

RETURN TO THE PROOF OF COROLLARY6:

We have: $\forall n \geq 1 \left(\frac{\ln(p_{n+1})}{\ln(p_n)}\right)^n < \left(1+\frac{1}{n}\right)^n \leq sup_{n\geq 1}\left(1+\frac{1}{n}\right)^n = e$

Corollary7: $\forall n \geq e$ we have: $\left(\frac{\ln(p_{n+1})}{\ln(p_n)}\right)^n < n\ln(n)$

Remark: we have: $e = 2.7182881 \ldots$ so:$\forall n$ integer $\geq e \Leftrightarrow \forall\, n$ integer ≥ 3

Proof: (of corollary 7)

*Consider the function $v(t) = t\ln(t) - e$

*$v(t) = 0 \Leftrightarrow t = e$

*$v'(t) = \ln(t) + 1$

*$v'(t) = 0 \Leftrightarrow t = e^{-1}$

*v Strictly increasing $\Leftrightarrow v'(t) > 0 \Leftrightarrow \ln(t) > -1 \Leftrightarrow t > e^{-1}$

*So: $e > e^{-1} \Rightarrow v$ is strictly increasing for $t \geq e \Rightarrow v(t) = t\ln(t) - e \geq e\ln(e) - e = e - e = 0$

*So, by corollary 4: $\forall n \geq e$: $\left(\frac{\ln(p_{n+1})}{\ln(p_n)}\right)^n < e \leq n\ln(n)$

Remark: We have: $2\ln(2) - \left(\frac{\ln(p_3)}{\ln(p_2)}\right)^2 = 2\ln(2) - \left(\frac{\ln(5)}{\ln(3)}\right)^2 = -0.0786 \ldots < 0$

2. PROOF OF THE SINHA CONJECTURE:

Corlollary8: (Nilotpal Kanti Sinha conjecture [19]) we have:

$$\forall n > 4\ \ p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1$$

Proof: (of corollary 8)

Corollary 8 will be deduced from the claims below.

Claim13: $\forall n \geq 1 \exists c_n \in]0, \frac{\ln(p_n)}{n}[:\ e^{\frac{\ln(p_n)}{n}} = 1 + \frac{\ln(p_n)}{n} + \frac{e^{c_n}}{2}\left(\frac{\ln(p_n)}{n}\right)^2$

Proof: (of claim13)

*Writing proposition 12 for the function $f(t) = e^t$ on $[0, \frac{\ln(p_n)}{n}]$, we have:

$\exists c_n \in]0, \frac{\ln(p_n)}{n}[$ such that: $e^{\frac{\ln(p_n)}{n}} = 1 + \frac{\ln(p_n)}{n} + \frac{e^{c_n}}{2}\left(\frac{\ln(p_n)}{n}\right)^2$

Claim14: We have: $\frac{p_n}{n} < \ln(p_n) - 1\ \forall n \geq 701$

Proof: (of claim14)

*By the Dursat theorem (the assertion (i) of proposition 4), we have:

$$\forall p_n \geq 5393\ \ \frac{p_n}{\ln(p_n) - 1} \leq \pi(p_n) = n$$

*5393 being a prime integer with $\pi(5393) = 701$, the result follows.

Claim15: We have: (i) the function $f(t) = \sqrt{t} - 4\ln(t) + 2$ is increasing for $t \geq 64$ and decreasing for $0 < t \leq 64$

(ii)The continuous function $f(t) = \sqrt{t} - 4\ln(t) + 2$ has only two zeros: $1 < \alpha < 64$ and $64 < \beta < 535$

(iii)$\forall t \geq 535\ \sqrt{t} - 4\ln(t) + 2 \geq 0$

(iv)The continuous function $g(t) = \sqrt{t} - (\ln(t))^2 + \ln(t)$ is increasing in $]535, +\infty[$ and has only one zero $\delta \in]535, 3347[$

(v)$\forall t \geq 3347\ \sqrt{t} \geq (\ln(t))^2 - \ln(t)$

Proof: of (claim15)

(i) *We have: $f'(t) = \frac{1}{2\sqrt{t}} - \frac{4}{t} = \frac{\sqrt{t}-8}{2t}$

*f Increasing $\Leftrightarrow t \geq 64 = 8^2$ and f decreasing $\Leftrightarrow t \leq 64 = 8^2$

(ii)*We have: $g(1) = 3 > 0$ and $g(64) = -6.6 \ldots < 0$

*So, by the intermediate value theorem: $\exists \alpha \in]1,64[$ such that $f(\alpha) = 0$

*So: f being strictly decreasing on] 1, 64[: α is the single zero in this interval.

*We have:$f(64) = -6.6 \ldots < 0$ and $f(535) = 0.001.. > 0$

*So, by the intermediate value theorem: $\exists \beta \in]64{,}535[$ such that $f(\beta) = 0$

*So: f being strictly increasing on] 64, $+\infty$[: β is the single zero in this interval.

(iii)So: $\forall t \geq 535 > \beta\ \ f(t) \geq f(\beta) = 0$

(iv)*We have: $g'(t) = \frac{1}{2\sqrt{t}} - \frac{2\ln(t)}{t} + \frac{1}{t} = \frac{\sqrt{t} - 4\ln(t) + 2}{2t} > 0\ for\ t \geq 535$ (by the assertion (iii) of claim 16)

*We have: $g(535) = -10..05 \ldots < 0\ and\ g(3347) = 0.1 \ldots > 0$

*So: by the intermediate value theorem:$\exists \delta \in]535{,}3347[\ g(\delta) = 0$, which is the sole zero because $g$is strictly increasing in this interval.

(v)So: $\forall t \geq 3347 > \delta\ \ g(t) = \sqrt{t} - (\ln(t))^2 + \ln(t) > g(\delta) = 0$

Claim17: $\forall n \geq 701\ \ \frac{(\ln(p_n))^2(\ln(p_n)-1)}{n} < 1$

Proof: (of claim17)

Remark: 3347 is a prime integer with $\pi(3347) = 457$

*By claim 14, we have: $\frac{p_n}{n(\ln(p_n)-1)} < 1$ for $n \geq 701$

*By the assertion (v) of claim 15, we have: for $p_n \geq 3347$ i. e. $n \geq 457$: $p_n \geq (\ln(p_n))^2(\ln(p_n) - 1)^2$

*So: $1 > \frac{p_n}{n(\ln(p_n)-1)} > \frac{(\ln(p_n))^2(\ln(p_n)-1)}{n}$ for $n \geq 701$

RETURN TO THE PROOF OF COROLLARY 8

*By the theorem and claim13, we have successively:

$$\forall n \geq 1\ \ p_{n+1}^{\frac{1}{n+1}} < p_n^{\frac{1}{n}} \Rightarrow p_{n+1} < p_n^{\frac{n+1}{n}} = p_n p_n^{\frac{1}{n}} = p_n e^{\frac{\ln(p_n)}{n}} = p_n\left(1 + \frac{\ln(p_n)}{n} + \frac{e^{c_n}}{2}\left(\frac{\ln(p_n)}{n}\right)^2\right)$$

*But, by corollary 3: $0 < c_n < \frac{\ln(p_n)}{n} < \ln(2) \Rightarrow e^{c_n} < e^{\ln(2)} = 2$

*That is, by claim14 and claim 16: $\forall n \geq 701$

$$p_{n+1} - p_n < \frac{p_n \ln(p_n)}{n}\left(1 + \left(\frac{\ln(p_n)}{n}\right)\right) < \ln(p_n)\,(\ln(p_n) - 1)\left(1 + \frac{\ln(p_n)}{n}\right)$$

$$= (\ln(p_n))^2 - \ln(p_n) + \frac{(\ln(p_n))^2(\ln(p_n)-1)}{n} < (\ln(p_n))^2 - \ln(p_n) + 1$$

Remark: We have showed:$\forall n \geq 701$: $p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1$, and this inequality can be verified for $5 \leq n \leq 700$. So, we have well: $\forall n \geq 5\ p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1$.

3. PROOF OF THE ANDRICA CONJECTURE

Corollary9: (Andrica conjecture)[1] we have: $\forall n \geq 1$

(i) $p_{n+1} - p_n < 2\sqrt{p_n}$

(ii) $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$

(iii) $\lim_{n \to +\infty} \frac{p_{n+1}}{p_n} = 1$

(iv) $\forall n \geq 2 \exists!\ \delta_n \in]\frac{1}{2}, 1[$ such that $\begin{cases} \forall x \in [\frac{1}{2}, \delta_n[\ \ p_{n+1}^x - p_n^x < 1 \\ p_{n+1}^{\delta_n} - p_n^{\delta_n} = 1 \end{cases}$

Proof: (of corollary 10)

(i)The proof of the assertion (i) of corollary 9 will be deduced from the claims below.

Claim17: The function $\theta(t) = \sqrt{t} + 1 - 2\ln(t)$ is strictly increasing for $t \geq 16$ and strictly decreasing for $0 < t \leq 16$.

Proof: (of claim17)

*We have: $\theta'(t) = \frac{1}{2\sqrt{t}} - \frac{2}{t} = \frac{\sqrt{t}-4}{2t}$

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*θ Strictly increasing $\Leftrightarrow \theta'(t) > 0 \Leftrightarrow \sqrt{t} - 4 > 0 \Leftrightarrow t > 16$
*θ Strictly decreasing $\Leftrightarrow \theta'(t) < 0 \Leftrightarrow \sqrt{t} - 4 < 0 \Leftrightarrow t < 16$
Claim18: the function $\theta(t) = \sqrt{t} + 1 - 2\ln(t)$ has only two zeros: $1 < \alpha < 16$ $and 16 < \beta < e^4$
Proof: (of claim18)
*θ is continuous on$]0, +\infty[$
*We have:$\theta(1) = 2 > 0$ and $\theta(16) = 5 - 8\ln(2) = -0.54 \ldots < 0$
*We have:$\theta(16) = -0.54 \ldots$ and $\theta(e^4) = \sqrt{e^4} + 1 - 2\ln(e^4) = e^2 + 1 - 8 = 0.38 \ldots > 0$
*So, by the intermediate value theorem: $\exists\alpha \in]1,16[$ such that: $\theta(\alpha) = 0$ and $\exists\beta \in]16, e^4[$ such that $\theta(\beta) = 0$
*θ being, by claim15, strictly decreasing on$]1, 16[$and strictly increasing on$]16, e^4[$: α, β are the sole zeros.
Claim19: $\forall t > e^4$ we have: $\sqrt{t} + 1 - 2\ln(t) > 0$
Proof: (of claim19)
By claim17 and claim 18, we have: $t > e^4 > \beta > 16 \Rightarrow \theta(t) = \sqrt{t} + 1 - 2\ln(t) > \theta(\beta) = 0$
Claim20: $\forall t > e^4$ we have: $(\ln(t))^2 - \ln(t) + 1 < 2\sqrt{t}$
Proof: (of claim20)
*Consider the derivable function: $\varphi(t) = 2\sqrt{t} - (\ln(t))^2 + \ln(t) - 1$
*By claim8, we have: $\varphi'(t) = \frac{1}{\sqrt{t}} + \frac{1}{t} - 2\frac{\ln(t)}{t} = \frac{\sqrt{t}+1-2\ln(t)}{t} > 0$ $\forall t > e^4 > \beta > 16$
*So: $\forall t > e^4$: $\varphi(t) = 2\sqrt{t} - (\ln(t))^2 + \ln(t) - 1 > \varphi(e^4) = 2e^2 - 16 + 4 - 1 = 1.77 \ldots > 0$

RETURN TO THE PROOF OF THE ASSERTION (i) OF COROLLARY 9:

*By corollary 8 and claim 20, we have:

$$\forall n \geq 701\ p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1 < 2\sqrt{p_n}$$

*So: $\forall n \geq 701\ p_{n+1} - p_n < 2\sqrt{p_n}$
Remark: We have: $p_n \geq e^4 \Rightarrow p_n \geq 59 \Rightarrow n \geq 17$ $(\pi(59) = 17)$
(ii)*We have: $\frac{2\sqrt{p_n}}{\sqrt{p_{n+1}}+\sqrt{p_n}} < 1 \Leftrightarrow 2\sqrt{p_n} < \sqrt{p_{n+1}} + \sqrt{p_n} \Leftrightarrow \sqrt{p_n} < \sqrt{p_{n+1}}$ (always true)
*So, by the assertion (i) of corollary 9, we have:
$p_{n+1} - p_n < 2\sqrt{p_n} \Rightarrow (\sqrt{p_{n+1}} - \sqrt{p_n})(\sqrt{p_{n+1}} + \sqrt{p_n}) < 2\sqrt{p_n}$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{2\sqrt{p_n}}{\sqrt{p_{n+1}}+\sqrt{p_n}} < 1$
Remark: We have showed here that:$\forall n \geq 701$: $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, and this relation was confirmed numerically for $1 \leq n \leq 700$. So we have: $\forall n \geq 1$ $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$
(iii)*By the assertion (ii) of corollary 9, we have:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \Rightarrow \sqrt{p_{n+1}} < \sqrt{p_n} + 1 \Rightarrow 1 < \sqrt{\frac{p_{n+1}}{p_n}} < 1 + \frac{1}{\sqrt{\mathrm{p_n}}}$$

*The result follows by the squeeze theorem.
(iv)*Consider, on $[\frac{1}{2}, 1]$ for $n \geq 2$ the continuous function $f_n(x) = p_{n+1}^x - p_n^x - 1$
**We have, by the assertion (ii) of corollary 9: $f_n\left(\frac{1}{2}\right) = p_{n+1}^{\frac{1}{2}} - p_n^{\frac{1}{2}} - 1 < 0$
**We have: $f_n(1) = p_{n+1} - p_n - 1 > 0$ (because $n \geq 2$)
*So, by the intermediate value theorem, $\exists\delta_n \in]\frac{1}{2}, 1[$ such that: $p_{n+1}^{\delta_n} - p_n^{\delta_n} = 1$
*δ_n is the sole real, in $]\frac{1}{2}, 1[$, having this property, because:
$\frac{p_{n+1}^x}{p_n^x} > 1 > \frac{\ln(p_n)}{\ln(p_{n+1})} \Rightarrow f_n'(x) = p_{n+1}^x \ln(p_{n+1}) - p_n^x \ln(p_n) > 0 \Rightarrow f_n$ is strictly increasing in $]\frac{1}{2}, 1[$.
*Suppose contrarily that $\exists x \in]\frac{1}{2}, \delta_n[$ such that: $p_{n+1}^x - p_n^x > 1$
*Because: $p_{n+1}^{\frac{1}{2}} - p_n^{\frac{1}{2}} < 1$, by the intermediate value theorem applied to the continuous function f_n on the interval $[\frac{1}{2}, x]$, we have: $\exists c \in]\frac{1}{2}, x[$ such that $f_n(x) = p_{n+1}^c - p_n^c - 1 = 0$ with $c \neq \delta_n$
*But this is impossible, because δ_n is the sole real, in $]\frac{1}{2}, 1[$ such that $p_{n+1}^{\delta_n} - p_n^{\delta_n} = 1$

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*So, $\forall n \geq 2 \exists ! \delta_n \in]\frac{1}{2}, 1[\begin{cases} \forall x \in [\frac{1}{2}, \delta_n[\ \ p_{n+1}^x - p_n^x < 1 \\ p_{n+1}^{\delta_n} - p_n^{\delta_n} = 1 \end{cases}$

4. PROOF OF THE CRAMER AND THE CRAMER-GRANVILLE CONJECTURES

Corollary10: We have:
(i)(The Weak Cramér conjecture [19], [39]): $p_{n+1} - p_n = O((\ln(p_n)))^2$,
(i) (The Strong Cramer conjecture [5])$\limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right) = 1$
(iii)(Cramer-Granville conjecture [15]) $\limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right) = 2e^{-\gamma}$, (Where $\gamma = 0.5772 \ldots$ denotes the Euler-Mascheroni constant) is not true.
Proof: (of corollary 10)
(i)*By corollary 8, we have: $\forall n > 4 \ \ p_{n+1} - p_n < (\ln(p_n))^2 - \ln(p_n) + 1 < (\ln(p_n))^2$
*So, using the Bachman-Landau symbol, we have: $p_{n+1} - p_n = O((\ln(p_n)))^2$
Remark:"$\limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right) = 1$" means that: "the gaps between consecutive primes are always small, and it quantifies asymptotically how small they can be" [19].
Remark: According to Sinha [19], the error $O((\ln(p_n))^2$ is optimum and it cannot be lowered.
(ii)*By corollary 8, we have:

$$\forall n \geq 5 \ p_{n+1} - p_n < (\ln(p_n))^2 + 1 - \ln(p_n) < (\ln(p_n))^2$$

*That is: $\frac{p_{n+1}-p_n}{(\ln(p_n))^2} < 1$
*So: $\limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right) \leq 1$
*Suppose contrarily that: $\forall \varphi$ bijection:$\mathbb{N}^* \to \mathbb{N}^* \ \exists \epsilon > 0 \forall N \exists n \geq N \ (1-\epsilon)\left(\ln\left(p_{\varphi(n)}\right)\right)^2 \geq p_{\varphi(n)+1} - p_{\varphi(n)}$
*Let: $n_N = \min\left(\left\{n \geq N, (1-\epsilon)\left(\ln\left(p_{\varphi(n)}\right)\right)^2 \geq p_{\varphi(n)+1} - p_{\varphi(n)}\right\}\right)$
*We have: $(1-\epsilon)\left(\ln\left(p_{\varphi(n_N)}\right)\right)^2 \geq p_{\varphi(n_N)+1} - p_{\varphi(n_N)}$ and $(1-\epsilon)\left(\ln\left(p_{\varphi(n_N-1)}\right)\right)^2 < p_{\varphi(n_N-1)+1} - p_{\varphi(n_N-1)}$
Remark: if $n_N = \min\left(\left\{n \geq N, (1-\epsilon)\left(\ln\left(p_{\varphi(n)}\right)\right)^2 \geq p_{\varphi(n)+1} - p_{\varphi(n)}\right\}\right) = N$, we have:
$\forall n \leq N-1 \ (1-\epsilon)\left(\ln\left(p_{\varphi(n)}\right)\right)^2 < p_{\varphi(n)+1} - p_{\varphi(n)}$
*So: $1 - \epsilon \leq \lim_{N\to+\infty} \frac{p_{\varphi(n_N-1)+1} - p_{\varphi(n_N-1)}}{\left(\ln\left(p_{\varphi(n_N-1)}\right)\right)^2} = \lim_{N\to+\infty} \frac{p_{\varphi(n_N)+1} - p_{\varphi(n_N)}}{\left(\ln\left(p_{\varphi(n_N)}\right)\right)^2} \leq 1 - \epsilon$
*That is: $\lim_{N\to+\infty} \frac{p_{\varphi(n_N)+1} - p_{\varphi(n_N)}}{\left(\ln\left(p_{\varphi(n_N)}\right)\right)^2} = 1 - \epsilon$
* But I have deduced in [14] (See also [10]), from the Riemann Hypothesis, the twin primes conjecture which says that:$\exists \psi: \mathbb{N}^* \to \mathbb{N}^*$ a bijection, such that:$\forall n \in \mathbb{N}^* \ p_{\psi(n)+1} - p_{\psi(n)} = 2$. So: $\lim_{n\to+\infty} \frac{p_{\psi(n)+1} - p_{\psi(n)}}{\left(\ln\left(p_{\psi(n)}\right)\right)^2} = 0$
*So writing our absurd hypothesis, in particular for $\varphi = \psi$, we deduce that:$1 - \epsilon = 0$.
*But this is impossible, because it implies $(1-\epsilon)\left(\ln\left(p_{\varphi(n_N)}\right)\right)^2 = 0 \geq p_{\varphi(n_N)+1} - p_{\varphi(n_N)} > 0$
*So, our starting absurd hypothesis is false,
* So its negation "$\exists \varphi$ bijection:$\mathbb{N}^* \to \mathbb{N}^* \ \forall \epsilon > 0 \exists N \forall n \geq N \ 1 - \epsilon < \frac{p_{\varphi(n)+1} - p_{\varphi(n)}}{\left(\ln\left(p_{\varphi(n)}\right)\right)^2} < 1$" is true.
*That is: $\limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right) = 1$ (limsup being the greatest adherent point)
(iii)The Cramer-Granville is not true because: $2e^{-\gamma} = 1.1229 \ldots > 1 = \limsup_{n\to+\infty}\left(\frac{p_{n+1}-p_n}{(\ln(p_n))^2}\right)$ by the assertion (ii) of corollary 11.
Remark: (i) By the twin primes conjecture (See [14] and [10]) which says that:$\exists \varphi: \mathbb{N}^* \to \mathbb{N}^*$ a bijection, such that:$\forall n \in \mathbb{N}^* \ p_{\varphi(n)+1} - p_{\varphi(n)} = 2$, we have: $\liminf_{n\to+\infty} \frac{p_{n+1}-p_n}{(\ln(p_n))^2} = 0$ (liminf being the smallest adherent value).
(ii) In 1931, E.Westzynthius showed, in [22], that: $\limsup_{n\to+\infty} \frac{p_{n+1}-p_n}{\ln(p_n)} = +\infty$. So: $\exists \varphi: \mathbb{N}^* \to \mathbb{N}^*$ a bijection such that: $\lim_{n\to+\infty} p_{\varphi(n)+1} - p_{\varphi(n)} = +\infty$.

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