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INTERVAL VALUED (α,β)-FUZZY H_v-IDEALS OF H_v-RINGS

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Abstract

In this paper we introduce the concept of an interval valued (α,β)-fuzzy H_v-ideal of an H_v-ring, which is a generalization of a fuzzy H_v-ideal of an H_v-ring. Also we introduce interval valued $(\in,\in\vee q)$-fuzzy H_v-ideal and some of their properties.

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I. Introduction

The algebraic hyperstructures are a suitable generalization of the classical algebraic structures which was first initiated by Marty [12]. Since then, hundreds of papers and several books have been written on this topic. A short review of which appears in [14]. A recent book on hyperstructures [15] points out their applications in geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Vougiouklis [22] introduced a new class of hyperstructures so-called H_v-structure, and Davvaz [3] surveyed the theory of H_v-structures. The H_v-structures are hyperstructures where equality is replaced by non-empty intersection.

The concept of fuzzy sets was first introduced by Zadeh [13] and then fuzzy sets have been used in the reconsideration of classical mathematics. In particular, the notion of fuzzy subgroup was defined by Rosenfeld [1] and its structure was thereby investigated. Liu [23] introduced the notions of fuzzy subrings and ideals. Using the notion of "belongingness$(\in)$" and "quasi-coincidence(q)" of fuzzy points with fuzzy sets, the concept of (α,β)-fuzzy subgroup where α,β are any two of $\{\in,q,\in\vee q,\in\wedge q\}$ with $\alpha\neq\in\wedge q$ was introduced in [17]. The most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in,\in\vee q)$-fuzzy subgroups, the detailed study of which may be found in [19]. The concept of an $(\in,\in\vee q)$-fuzzy subring and ideal of a ring have been introduced in [18] and the concept of $(\in,\in\vee q)$-fuzzy subnear-ring and ideal of a near-ring have been introduced in [2]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now studied both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by many authors. In [4, 5, 7], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy H_v-subgroups, fuzzy H_v-ideals and fuzzy H_v-submodules, which are generalizations of the concepts of Rosenfeld's fuzzy subgroups, fuzzy ideals and fuzzy submodules. The concept of a fuzzy H_v-ideal and H_v-subring has been studied further in [6, 8]. Davvaz [9] introduced the concept of an interval valued (α,β)-fuzzy H_v-submodule of H_v-module. This paper continues this line of research for interval valued (α,β)-fuzzy H_v-ideal of an H_v-ring.

The paper is organized as follows: In Section 2, we first recall some basic definitions and results of H_v-rings and H_v-ideals. In Section 3, we extend quasi-coincidence of fuzzy point in a fuzzy set to quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set. Since the concept of an $(\in,\in\vee q)$-fuzzy H_v-ideal generalizes that of an ordinary fuzzy H_v-ideal, some fundamental aspects of such $(\in,\in\vee q)$-fuzzy H_v-ideals will be discussed in Section 4. Also we extend the concept of a fuzzy H_v-subgroup with thresholds to the concept of interval valued fuzzy H_v-ideal with thresholds.

II. Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1. [11] Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x) \quad \forall x \in X$.

Definition 2.2. [20] Let G be a non–empty set and $* : G \times G \to \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of G. Where $A * B = \bigcup_{a \in A, b \in B} a * b, \ \forall A, B \subseteq G.$

The $*$ is called weak commutative if $x * y \cap y * x \neq \phi, \forall x, y \in G.$

The $*$ is called weak associative if $(x * y) * z \cap x * (y * z) \neq \phi, \forall x, y, z \in G.$

A hyperstructure $(G, *)$ is called an H_v-group if

(i) $*$ is weak associative.

(ii) $a * G = G * a = G, \quad \forall a \in G$ (Reproduction axiom).

Definition 2.3. [20] An H_v-ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the ring-like axioms:

(i) $(R, +)$ is an H_v-group, that is, $((x + y) + z) \cap (x + (y + z)) \neq \phi \quad \forall x, y \in R,$ $a + R = R + a = R \quad \forall a \in R;$

(ii) $(R, \cdot)$ is an H_v-semigroup;

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, that is, for all $x, y, z \in R$ $(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \phi,$ $((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \phi.$

Definition 2.4. [10] Let R be an H_v-ring. A nonempty subset I of R is called a left (resp., right) H_v-ideal if the following axioms hold:

(i) $(I, +)$ is an H_v-subgroup of $(R, +)$,

(ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).

Definition 2.5. [10] Let $(R, +, \cdot)$ be an H_v–ring and μ a fuzzy subset of R. Then μ is said to be a left (resp., right) fuzzy H_v-ideal of R if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in R,$

(2) For all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y),$

(3) For all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z),$

(4) $\mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\}$ respectively $\mu(x) \leq \inf\{\mu(z) : z \in x \cdot y\} \quad \forall x, y \in R.$

Definition 2.6. [20] Let μ be a fuzzy subset of R. If there exist a $t \in (0,1]$ and an $x \in R$ such that

$$\mu(y) = \begin{cases} t & if \ y = x \\ 0 & if \ y \neq x \end{cases}$$

Then μ is called a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.7. [20] Let μ be a fuzzy subset of R and x_t be a fuzzy point.

(1) If $\mu(x) \geq t$, then we say x_t belongs to μ, and write $x_t \in \mu$.

(2) If $\mu(x) + t > 1$, then we say x_t is quasi-coincident with μ, and write $x_t q \mu$.

(3) $x_t \in \vee q \mu \Leftrightarrow x_t \in \mu \ or \ x_t q \mu.$

(4) $x_t \in \wedge q \mu \Leftrightarrow x_t \in \mu \ and \ x_t q \mu.$

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ with $\alpha \neq \in \wedge q$, which was introduced by Bhakat and Das [9].

Definition 2.8. [20] Let R be an H_v–ring. A fuzzy subset μ of R is said to be an (α, β)–fuzzy left (right) H_v–ideals of R if for all $t, r \in (0,1],$

(1) $x_t \alpha \mu, y_r \alpha \mu \Rightarrow z_{t \wedge r} \beta \mu \quad \forall z \in x + y,$

(2) $x_t \alpha \mu, a_r \alpha \mu \Rightarrow y_{t \wedge r} \beta \mu$ for some $y \in R$ with $x \in a + y,$

(3) $x_t \alpha \mu, a_r \alpha \mu \Rightarrow z_{t \wedge r} \beta \mu$ for some $z \in R$ with $x \in z + a,$

(4) $y_t \alpha \mu, x \in R \Rightarrow z_t \beta \mu, \forall z \in x.y$ $(x_t \alpha \mu, y \in R \Rightarrow z_t \beta \mu, \forall z \in x.y).$

By an interval number $\tilde{a}$ we mean an interval $\left[a^{-},a^{+}\right]$ where $0\le a^{-}\le a^{+}\le 1$. The set of all interval numbers is denoted by $D[0,1]$ [43]. We also identify the interval $[a,a]$ by the number $a\in[0,1]$.

For the interval numbers $\tilde{a}_i=\left[a_i^{-},a_i^{+}\right]\in D[0,1], i\in I,$ we define

$$\max\left\{\tilde{a}_i,\tilde{b}_i\right\}=\left[\max\left(a_i^{-},b_i^{-}\right),\max\left(a_i^{+},b_i^{+}\right)\right],$$

$$\min\left\{\tilde{a}_i,\tilde{b}_i\right\}=\left[\min\left(a_i^{-},b_i^{-}\right),\min\left(a_i^{+},b_i^{+}\right)\right],$$

$$\inf\tilde{a}_i=\left[\underset{i\in I}{\wedge}a_i^{-},\underset{i\in I}{\wedge}a_i^{+}\right],\sup\tilde{a}_i=\left[\underset{i\in I}{\vee}a_i^{-},\underset{i\in I}{\vee}a_i^{+}\right]$$

and put

$(1)\ \tilde{a}_1\le\tilde{a}_2\Leftrightarrow a_1^{-}\le a_2^{-}$ and $a_1^{+}\le a_2^{+}$,

$(2)\ \tilde{a}_1=\tilde{a}_2\Leftrightarrow a_1^{-}=a_2^{-}$ and $a_1^{+}=a_2^{+}$,

$(3)\ \tilde{a}_1<\tilde{a}_2\Leftrightarrow\tilde{a}_1\le\tilde{a}_2$ and $\tilde{a}_1\ne\tilde{a}_2$,

$(4)\ k\tilde{a}=\left[ka^{-},ka^{+}\right]$, whenever $0\le k\le 1$.

It is clear that $\left(D[0,1],\le,\vee,\wedge\right)$ is a complete lattice with $0=[0,0]$ as least element and $1=[1,1]$ as greatest element.

By an interval valued fuzzy set F on X we mean the set $F=\left\{\left(x,\left[\mu_F^{-}(x),\mu_F^{+}(x)\right]\right):x\in X\right\}$. Where μ_F^{-} and μ_F^{+} are fuzzy subsets of X such that $\mu_F^{-}(x)\le\mu_F^{+}(x)$ for all $x\in X$. Put $\tilde{\mu}_F(x)=\left[\mu_F^{-}(x),\mu_F^{+}(x)\right]$. Then $F=\left\{\left(x,\tilde{\mu}_F(x)\right):x\in X\right\}$, where $\tilde{\mu}_F:X\to D[0,1]$.

If A, B are two interval valued fuzzy subsets of X, then we define

$A\subseteq B$ if and only if for all $x\in X$, $\mu_A^{-}(x)\le\mu_B^{-}(x)$ and $\mu_A^{+}(x)\le\mu_B^{+}(x)$,

$A=B$ if and only if for all $x\in X$, $\mu_A^{-}(x)=\mu_B^{-}(x)$ and $\mu_A^{+}(x)=\mu_B^{+}(x)$.

Also, the union, intersection and complement are defined as follows: let A; B be two interval valued fuzzy subsets of X, then

$$A\cup B=\left\{\begin{array}{l}\left(x,\left[\max\left\{\mu_A^{-}(x),\mu_B^{-}(x)\right\},\max\left\{\mu_A^{+}(x),\mu_B^{+}(x)\right\}\right]\right)\\ :x\in X\end{array}\right\},$$

$$A\cap B=\left\{\begin{array}{l}\left(x,\left[\min\left\{\mu_A^{-}(x),\mu_B^{-}(x)\right\},\min\left\{\mu_A^{+}(x),\mu_B^{+}(x)\right\}\right]\right)\\ :x\in X\end{array}\right\},$$

$$A^c=\left\{\left(x,\left[\left\{1-\mu_A^{-}(x),1-\mu_A^{+}(x)\right\}\right]\right):x\in X\right\}.$$

III. Interval valued (α,β)-fuzzy H_v-ideals

The concept of Rosenfeld's fuzzy subgroups with interval valued membership functions was first introduced by Biswas in [26]. Davvaz applied this concept to the theory of fuzzy hyperstructures in [2, 11, 27]. In this section, we extend the quasicoincidence of fuzzy point in a fuzzy set to the quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set as follows:

We first call an interval valued fuzzy set F of an H_v-ring R of the form

$$\tilde{\mu}_F(y)=\begin{cases}\tilde{t}(\ne[0,0]) & y=x,\\ [0,0] & y\ne x,\end{cases}$$

a fuzzy interval value with support x and interval value $\tilde{t}$ and denote it by $U(x;\tilde{t})$. A fuzzy interval value $U(x;\tilde{t})$ is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set F, written as $U(x;\tilde{t})\in F$ (resp. $U(x;\tilde{t})qF$) if $\tilde{\mu}_F(x)\ge\tilde{t}$ (resp. $\tilde{\mu}_F(x)+\tilde{t}>[1,1]$). If $U(x;\tilde{t})\in F$ or (resp. and) $U(x;\tilde{t})qF$ then we write $U(x;\tilde{t})\in\vee q$ (resp. $\in\wedge q$) F. We use the symbol $\overline{\in\vee q}$ that means $\in\vee q$ does not hold. In what follows, R is an H_v-ring, and α and β are any one of $\in,q,\in\vee q$ or $\in\wedge q$ unless specified. Also, we emphasis here that $\tilde{\mu}_F(x)=\left[\mu_F^{-}(x),\mu_F^{+}(x)\right]$ must satisfy the following properties: $\left[\mu_F^{-}(x),\mu_F^{+}(x)\right]<[0.5,0.5]$ or $[0.5,0.5]\le\left[\mu_F^{-}(x),\mu_F^{+}(x)\right]$ for all $x\in R$.

We now formulate the following definition.

Definition 3.1. An interval valued fuzzy set F of R is called an interval valued (α,β)-fuzzy H_v-ideal of R if for all $t,r\in(0,1]$ and $x,y\in R,$ the following conditions hold:

$(I)U(x;\tilde{t})\alpha F$ and $U(y;\tilde{r})\alpha F$ imply $U(z;\min\{\tilde{t},\tilde{r}\})\beta F$ for all $z\in x+y,$

$(II)U(x;\tilde{t})\alpha F$ and $U(a;\tilde{r})\alpha F$ imply $U(y;\min\{\tilde{t},\tilde{r}\})\beta F$ for some $y\in R$ with $x\in a+y,$

$(III)U(x;\tilde{t})\alpha F$ and $U(a;\tilde{r})\alpha F$ imply $U(z;\min\{\tilde{t},\tilde{r}\})\beta F$ for some $z\in R$ with $x\in z+a,$ $(IV)U(x;\tilde{r})\alpha F$ imply $U(z;\tilde{r})\beta F$ for all $z\in c.x.$

Let F be an interval valued fuzzy set of R such that $\tilde{\mu}_F(x)\le[0.5,0.5],$ for all $x\in R.$ Let $x\in R$ and $t\in(0,1]$ be such that $U(x;\tilde{t})\in\wedge qF.$ Then $\tilde{\mu}_F(x)\ge\tilde{t}$ and $\tilde{\mu}_F(x)+\tilde{t}>[1,1].$ It follows that $[1,1]<\tilde{\mu}_F(x)+\tilde{t}\le\tilde{\mu}_F(x)+\tilde{\mu}_F(x)=2\tilde{\mu}_F(x).$ This implies that $\tilde{\mu}_F(x)>[0.5,0.5].$ Hence, $\{U(x;\tilde{t}):U(x;\tilde{t})\in\wedge qF\}=\phi,$ and consequently, the case $\alpha=\in\wedge q$ in Definition 3.1 can be omitted.

We now have the following properties:

(1) Every interval valued $(\in\vee q,\in\vee q)$-fuzzy H_v-ideal of R is an interval valued $(\in,\in\vee q)$-fuzzy H_v-ideal of R.

(2) Every interval valued $(\in,\in)$-fuzzy H_v-ideal of R is an interval valued $(\in,\in\vee q)$-fuzzy H_v-ideal of R.

(3) For any subset A of R, χ_A is an interval valued $(\in,\in\vee q)$-fuzzy H_v-ideal of R if and only if A is an H_v-ideal of R.

(4) Let F be a non-zero interval valued (α,β)-fuzzy H_v-ideal of R. Then the set $U(F;[0,0])=\{x\in R:\tilde{\mu}_F(x)>[0,0]\}$ is an H_v-ideal.

Let F be an interval valued fuzzy set. For every $t\in[0,1],$ the set $U(F;\tilde{t})=\{x\in H:\tilde{\mu}_F(x)\ge\tilde{t}\}$ is called the interval valued level subset of F. An interval valued fuzzy set F of an H_v-ring R is said to be proper if ImF has at least two elements. Two interval valued fuzzy sets are said to be equivalent if they have same family of interval valued level subsets. Otherwise, they are said to be non-equivalent.

We note here that for a proper H_v-ideal of the ring R, the proper interval valued $(\in,\in)$-fuzzy H_v-ideal F of R with cardImF $\ge$ 3 can be expressed as the union of two proper non-equivalent interval valued $(\in,\in)$-fuzzy H_v-ideal of R.

IV. Interval valued $(\in,\in\vee q)$-fuzzy H_v-ideals

Some fundamental aspects of interval valued $(\in,\in\vee q)$-fuzzy H_v-ideals of an H_v-ring R is discussed in this section. We first extend fuzzy H_v-ideals to interval valued fuzzy H_v-ideals of R. We start with the following definition:

Definition 4.1. An interval valued fuzzy set F of R is said to be an interval valued fuzzy H_v-ideal of R if it satisfies the following conditions:

$(I)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}\le\inf\{\tilde{\mu}_F(z):z\in x+y\}$ for all $x,y\in R,$

(II) for all $x,a\in R,$ there exists $y\in R$ such that $x\in a+y$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x)\}\le\tilde{\mu}_F(y),$

(III) for all $x,a\in R,$ there exists $z\in R$ such that $x\in z+a$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x)\}\le\tilde{\mu}_F(z),$

$(IV)\tilde{\mu}_F(x)\le\inf\{\tilde{\mu}_F(z):z\in c.x\},$ for all $x\in R.$

We now proceed to characterize the interval valued fuzzy H_v-ideals by using their level H_v-ideals.

Theorem 4.2. An interval valued fuzzy set F of R is an interval valued fuzzy H_v-ideal of R if and only if for any $[0,0]<\tilde{t}\le[1,1],U(F;\tilde{t})(\ne\phi)$ is an H_v-ideal of R.

Now, we introduce the following concept:

Definition 4.3. An interval valued fuzzy set F of R is said to be an interval valued $(\in,\in\vee q)$-fuzzy H_v-

ideal of R if for all $t,r\in(0,1]$ and $x,y\in R,$ the following conditions are satisfied:

$(I)U(x;\tilde{t})\in F$ and $U(y;\tilde{r})\in F$ imply $U(z;\min\{\tilde{t},\tilde{r}\})\in\vee qF$ for all $z\in x+y,$

$(II)U(x;\tilde{t})\in F$ and $U(a;\tilde{r})\in F$ imply $U(y;\min\{\tilde{t},\tilde{r}\})\in\vee qF$ for some $y\in R$ with $x\in a+y,$ $(III)U(x;\tilde{t})\in F$ and $U(a;\tilde{r})\in F$ imply $U(z;\min\{\tilde{t},\tilde{r}\})\in\vee qF$ for some $z\in R$ with $x\in z+a,$ $(IV)U(x;\tilde{r})\in F$ imply $U(z;\tilde{r})\in\vee qF$ for all $z\in c\cdot x.$

We first observe that if F is an interval valued fuzzy H_v-ideal of R according to Definition 4.1, then, F is an interval valued $(\in,\in\vee q)$-fuzzy H_v-ideal of R, by Definition 4.3.

We now formulate the following theorem:

Theorem 4.4. The conditions $(I)-(IV)$ in Definition 4.3 are equivalent to the following corresponding conditions:

$(i)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y),[0.5,0.5]\}\le\inf\{\tilde{\mu}_F(z):z\in x+y\},$ for all $x,y\in R,$

(ii) For all $x,a\in R,$ there exists $y\in R$ such that $x\in a+y$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x),[0.5,0.5]\}\le\tilde{\mu}_F(y),$

(iii) For all $x,a\in R,$ there exists $z\in R$ such that $x\in z+a$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x),[0.5,0.5]\}\le\tilde{\mu}_F(z),$

$(iv)\min\{\tilde{\mu}_F(x),[0.5,0.5]\}\le\inf\{\tilde{\mu}_F(z):z\in c\cdot x\},$ for all $x\in R.$

Proof. $(I)\Rightarrow(i)$ Suppose that $x,y\in R.$ Then, we consider the following cases:

$(a)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}<[0.5,0.5],$

$(b)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}\ge[0.5,0.5].$

Case (a): Assume that there exists $z\in x+y$ such that $\tilde{\mu}_F(z)<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y),[0.5,0.5]\}.$ Then, we have $\tilde{\mu}_F(z)<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}.$ Now, choose t such that $\tilde{\mu}_F(z)<\tilde{t}<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}.$ Then, we can easily see that $U(x;\tilde{t})\in F$ and $U(y;\tilde{t})\in F$ but $U(z;\tilde{t})\overline{\in\vee q}F.$ However, this contradicts (IV).

Case (b): Assume that $\tilde{\mu}_F(z)<[0.5,0.5]$ for some $z\in x+y.$ Then, $U(x;[0.5,0.5])\in F$ and $U(y;[0.5,0.5])\in F,$ but $U(z;[0.5,0.5])\overline{\in\vee q}F,$ again this is a contradiction. Hence, (iv) holds.

$(II)\Rightarrow(ii)$ Suppose that $x,a\in R.$ We now consider the following two cases:

$(a)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(a)\}<[0.5,0.5],$

$(b)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(a)\}\ge[0.5,0.5].$

Case (a): Assume that for any $y\in R$ with $x\in a+y.$ Then, we have $\tilde{\mu}_F(y)<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(a)\}.$ Choose t such that $\tilde{\mu}_F(y)<\tilde{t}<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(a)\}$ and $\tilde{t}+\tilde{\mu}_F(y)<[1,1].$ Then, $U(x;\tilde{t})\in F$ and $U(a;\tilde{t})\in F,$ but $U(y;\tilde{t})\overline{\in\vee q}F,$ which contradicts (II).

Case (b): Assume that for all $y\in R$ with $x\in a+y.$ Then, we have $\tilde{\mu}_F(y)<\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(a),[0.5,0.5]\}.$ Thus, $U(x;[0.5,0.5])\in F$ and $U(y;[0.5,0.5])\in F,$ but $U(y;[0.5,0.5])\overline{\in\vee q}F,$ which contradicts (II). Hence, (ii) holds.

$(III)\Rightarrow(iii)$ The proof is similar to $(II)\Rightarrow(ii)$ and is consequently omitted.

$(IV)\Rightarrow(iv)$ Suppose that $x\in R.$ We consider the following cases:

$(a)\mu_F(x)<[0.5,0.5],$

$(b)\tilde{\mu}_F(x)\ge[0.5,0.5].$

Case (a): Assume that there exists $z \in c \cdot x$ such that $\tilde{\mu}_F(z) < \min\{\tilde{\mu}_F(x), [0.5, 0.5]\}$. Then, this implies that $\tilde{\mu}_F(z) < \tilde{\mu}_F(x)$. Choose t such that $\tilde{\mu}_F(z) < \tilde{t} < \tilde{\mu}_F(x)$. Thus, we obtain $U(x; \tilde{t}) \in F$ but $U(z; \tilde{t}) \overline{\in \vee q} F$. This contradicts (IV).

Case (b): Assume that $\tilde{\mu}_F(z) < [0.5, 0.5]$ for some $z \in c \cdot x$. Then $U(x; [0.5, 0.5]) \in F$ but $U(z; [0.5, 0.5]) \overline{\in \vee q} F$, again a contradiction. Hence, (iv) holds.

$(i) \Rightarrow (I)$ Let $U(x; \tilde{t}) \in F$ and $U(y; \tilde{r}) \in F$. Then $\tilde{\mu}_F(x) \geq \tilde{t}$ and $\tilde{\mu}_F(y) \geq \tilde{r}$. For every $z \in x + y$, we have $\tilde{\mu}_F(z) \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]\} \geq \min\{\tilde{t}, \tilde{r}, [0.5, 0.5]\}$. If $\min\{\tilde{t}, \tilde{r}\} > [0.5, 0.5]$, then $\tilde{\mu}_F(z) \geq [0.5, 0.5]$. This implies that, $\tilde{\mu}_F(z) + \min\{\tilde{t}, \tilde{r}\} > [1, 1]$. If $\min\{\tilde{t}, \tilde{r}\} \leq [0.5, 0.5]$, then $\tilde{\mu}_F(z) \geq \min\{\tilde{t}, \tilde{r}\}$. Therefore, $U(z; \min\{\tilde{t}, \tilde{r}\}) \in \vee q F$, for all $z \in x + y$.

$(ii) \Rightarrow (II)$ Let $U(x; \tilde{t}) \in F$ and $U(a; \tilde{r}) \in F$. Then $\tilde{\mu}_F(x) \geq \tilde{t}$ and $\tilde{\mu}_F(a) \geq \tilde{r}$. Now, for some y with $x \in a + y$, we have $\tilde{\mu}_F(y) \geq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), [0.5, 0.5]\} \geq \min\{\tilde{t}, \tilde{r}, [0.5, 0.5]\}$. If $\min\{\tilde{t}, \tilde{r}\} > [0.5, 0.5]$, then $\tilde{\mu}_F(y) \geq [0.5, 0.5]$. This implies that $\tilde{\mu}_F(y) + \min\{\tilde{t}, \tilde{r}\} > [1, 1]$. If $\min\{\tilde{t}, \tilde{r}\} \leq [0.5, 0.5]$, then $\tilde{\mu}_F(y) \geq \min\{\tilde{t}, \tilde{r}\}$. Therefore, $U(y; \min\{\tilde{t}, \tilde{r}\}) \in \vee q F$. This shows that (II) holds.

$(iii) \Rightarrow (III)$ The proof is similar to $(ii) \Rightarrow (II)$ and we omit the proof.

$(iv) \Rightarrow (IV)$ Let $U(x; \tilde{r}) \in F$. Then $\tilde{\mu}_F(x) \geq \tilde{r}$. For every $z \in c \cdot x$, we have $\tilde{\mu}_F(z) \geq \min\{\tilde{\mu}_F(x), [0.5, 0.5]\} \geq \min\{\tilde{r}, [0.5, 0.5]\}$. If $\tilde{r} > [0.5, 0.5]$, then $\tilde{\mu}_F(z) \geq [0.5, 0.5]$. This implies that $\tilde{\mu}_F(z) + \tilde{r} > [1, 1]$. If $\tilde{r} \leq [0.5, 0.5]$, then $\tilde{\mu}_F(z) \geq \tilde{r}$. Therefore, $U(z; \tilde{r}) \in \vee q F$, for all $z \in c \cdot x$.

By Definition 4.3 and Theorem 4.4, we obtain the following corollary:

Corollary 4.5. An interval valued fuzzy set F of R is an interval valued $(\in, \in \vee q)$-fuzzy H_v-ideal of R if and only if all the conditions in Theorem 4.4 hold.

We now characterize the interval valued $(\in, \in \vee q)$-fuzzy H_v-ideals by using their level -ideals.

Theorem 4.6. Let F be an interval valued $(\in, \in \vee q)$-fuzzy H_v-ideal of R. Then for all $[0,0] < \tilde{t} \leq [0.5, 0.5], U(F; \tilde{t})$ is an empty set or an H_v-ideal of R. Conversely, if F is an interval valued fuzzy set of R such that $U(F; \tilde{t}) (\neq \phi)$ is an H_v-ideal of R for all $[0,0] < \tilde{t} \leq [0.5, 0.5], U(F; \tilde{t})$, then F is an interval valued $(\in, \in \vee q)$-fuzzy H_v-ideal of R.

Proof. Let F be an interval valued $(\in, \in \vee q)$ -fuzzy H_v-ideal of R and $[0,0] < \tilde{t} \leq [0.5, 0.5]$. If $x, y \in U(F; \tilde{t})$, then $\tilde{\mu}_F(x) \geq \tilde{t}$ and $\tilde{\mu}_F(y) \geq \tilde{t}$. Now we have

$$\inf\{\tilde{\mu}_F(z) : z \in x + y\} \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]\} \geq \min\{\tilde{t}, [0.5, 0.5]\} = \tilde{t}.$$

Therefore, for every $z \in x + y$, we have $\tilde{\mu}_F(z) \geq \tilde{t}$ or $z \in U(F; \tilde{t})$, and thereby $x + y \subseteq U(F; \tilde{t})$. Hence, for every $a \in U(F; \tilde{t})$, we have $a + U(F; \tilde{t}) \subseteq U(F; \tilde{t})$. Now, let $x, a \in U(F; \tilde{t})$. Then there exists $y \in R$

such that $x \in a+y$ and $\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), [0.5,0.5]\} \le \tilde{\mu}_F(y)$.

From $x, a \in U(F;\tilde{t})$, we have $\tilde{\mu}_F(x) \ge \tilde{t}$ and $\tilde{\mu}_F(a) \ge \tilde{t}$ and consequently, $\tilde{t} = \min\{\tilde{t}, \tilde{t}, [0.5,0.5]\} \le \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), [0.5,0.5]\} \le \tilde{\mu}_F(y)$.

Hence $y \in U(F;\tilde{t})$, and this leads to $U(F;\tilde{t}) \subseteq a + U(F;\tilde{t})$. Therefore $U(F;\tilde{t}) = a + U(F;\tilde{t})$. Similarly, we have $U(F;\tilde{t}) = U(F;\tilde{t}) + a$. Thus,

$$U(F;\tilde{t}) = a + U(F;\tilde{t}) = U(F;\tilde{t}) + a.$$

Let $c \in R$ and $x \in U(F;\tilde{t})$, Then $\tilde{\mu}_F(x) \ge \tilde{t}$. Now we have $\inf\{\tilde{\mu}_F(z) : z \in c \cdot x\} \ge \min\{\tilde{\mu}_F(x), [0.5,0.5]\} \ge \min\{\tilde{t}, [0.5,0.5]\} = \tilde{t}$.

Thus, for every $z \in c \cdot x$ we have $\tilde{\mu}_F(z) \ge \tilde{t}$ or $z \in U(F;\tilde{t})$, and whence $c \cdot x \subseteq U(F;\tilde{t})$. This shows that $U(F;\tilde{t})$ is indeed an H_v-ideal of R.

Conversely, let F be an interval valued fuzzy set of R such that $U(F;\tilde{t})(\ne \phi)$ is H_v-ideal of R for all $[0,0] < \tilde{t} \le [0.5,0.5]$, Then, for every $x, y \in R$, we can write $\tilde{\mu}_F(x) \ge \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5,0.5]\} = \tilde{t}_0$,

Thus we deduce that $x, y \in U(F;\tilde{t}_0)$, and so $x + y \subseteq U(F;\tilde{t}_0)$. Therefore, for every $z \in x + y$, we have $\tilde{\mu}_F(z) \ge \tilde{t}_0$. This implies that $\inf\{\tilde{\mu}_F(z) : z \in x+y\} \ge \tilde{t}_0$. Hence, in this way, we can verify condition (I) of Theorem 4.4. In order to verify the second condition, we put $\tilde{t}_1 = \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), [0.5,0.5]\}$, for every $x, a \in R$. Then, we have $x, a \in U(F;\tilde{t})$, thus, there exists $y \in U(F;\tilde{t}_1)$ so that $x \in a + y$. Since $y \in U(F;\tilde{t}_1), \tilde{\mu}_F(y) \ge \tilde{t}_1$ or $\tilde{\mu}_F(y) \ge \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), [0.5,0.5]\}$, the condition is verified. The third and fourth conditions can also be similarly verified. Therefore F is an interval valued $(\in, \in \vee q)$-fuzzy H_v-ideal of R.

A corresponding result can be similarly deduced when $U(F;t)$ is an H_v-ideal of R, for all $[0.5,0.5] < \tilde{t} \le [1,1]$.

Theorem 4.7. Let F be an interval valued fuzzy set of R. Then $U(F;\tilde{t})(\ne \phi)$ is an H_v-ideals of R for all $[0.5,0.5] < \tilde{t} \le [1,1]$ if and only if the following conditions are satisfied:

$$(I) \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} \le \inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z), [0.5,0.5]\} \\ : z \in x+y\end{matrix}\right\},$$

for all $x, y \in R$,

(II) For all $x, a \in R$, there exists $y \in R$ such that $x \in a + y$ and

$$\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x)\} \le \max\{\tilde{\mu}_F(y), [0.5,0.5]\},$$

(III) For all $x, a \in R$, there exists $z \in R$ such that $x \in z + a$ and

$$\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x)\} \le \max\{\tilde{\mu}_F(z), [0.5,0.5]\},$$

$$(IV) \tilde{\mu}_F(x) \le \inf\{\max\{\tilde{\mu}_F(z), [0.5,0.5]\} : z \in c \cdot x\},$$

for all $c \in R$.

Proof. Assume that $U(F;\tilde{t})$ is an H_v-ideals of R. If there exist $x, y, z \in R$ with $z \in x + y$ such that $\max\{\tilde{\mu}_F(z), [0.5,0.5]\} < \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} = \tilde{t}$, then $[0.5,0.5] < \tilde{t} \le [1,1]$, $\tilde{\mu}_F(z) < \tilde{t}, x, y \in U(F;\tilde{t})$. Since $x, y \in U(F;\tilde{t})$ and $U(F;\tilde{t})$ is an H_v-ideals of R, $x + y \subseteq U(F;\tilde{t})$ and $\tilde{\mu}_F(z) \ge \tilde{t}$, for all $z \in x + y$. This contradicts $\tilde{\mu}_F(z) < \tilde{t}$. Therefore, $\min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} \le \max\{\tilde{\mu}_F(z), [0.5,0.5]\}$ for all $x, y, z \in R$ with $z \in x + y$. This implies that

$\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}\le\inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z),[0.5,0.5]\}:\\ z\in x+y\end{matrix}\right\}$ for all $x,y\in R.$ Hence (i) holds.

Now, we assume that there exist $x_0,a_0\in R$ such that for all $y\in R$ with $x_0\in a_0+y.$ Then the following inequality holds: $\max\{\tilde{\mu}_F(y),[0.5,0.5]\}<\min\{\tilde{\mu}_F(a_0),\tilde{\mu}_F(x_0)\}=\tilde{t}.$ Thus, $[0.5,0.5]<\tilde{t}\le[1,1],x_0,a_0\in U(F;\tilde{t})$ and $\tilde{\mu}_F(y)<\tilde{t}.$ Since $x_0,a_0\in U(F;\tilde{t})$ and $U(F;\tilde{t})$ is an H_v-ideals, there exists $y_0\in U(F;\tilde{t})$ such that $x_0\in a_0+y_0.$ From $y_0\in U(F;\tilde{t}_0),$ we get $U(F;\tilde{t}_0)\ge\tilde{t},$ which contradicts $\tilde{\mu}_F(y_o)<\tilde{t}.$ Therefore for all $x,a\in R,$ there exists $y\in R$ such that $x\in a+y$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x)\}\le\max\{\tilde{\mu}_F(y),[0.5,0.5]\}.$ This demonstrates that (ii) holds.

The proof of the third condition is similar to the proof the second condition. If there exist $x,z,c\in R$ with $z\in c\cdot x$ such that $\max\{\tilde{\mu}_F(z),[0.5,0.5]\}<\tilde{\mu}_F(x)=\tilde{t},$ then $[0.5,0.5]<\tilde{t}\le[1,1],\tilde{\mu}_F(z)<\tilde{t},x\in U(F;\tilde{t}).$ Since $x\in U(F;\tilde{t})$ and $U(F;\tilde{t})$ is an H_v -ideal of R, $c\cdot x\subseteq U(F;\tilde{t})$ and $\tilde{\mu}_F(z)\ge\tilde{t},$ for all $z\in c\cdot x,$ which again contradicts $\tilde{\mu}_F(z)<\tilde{t}.$ Therefore $\tilde{\mu}_F(x)\le\max\{\tilde{\mu}_F(z),[0.5,0.5]\},$ for all $x,z,c\in R$ with $z\in c\cdot x.$ This leads to $\tilde{\mu}_F(x)\le\inf\{\max\{\tilde{\mu}_F(z),[0.5,0.5]\}:z\in c\cdot x\},$ for all $x,c\in R.$ Hence condition (iv) is verified.

Conversely, suppose that the conditions (i)–(iv) hold. We only need to show that $U(F;\tilde{t})$ is an H_v-ideal of R. For this purpose, we assume that $[0.5,0.5]<\tilde{t}\le[1,1]$ and $x,y\in U(F;\tilde{t})$ with $a\in R.$ Then $[0.5,0.5]<\tilde{t}\le\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y)\}$ $\le\inf\{\max\{\tilde{\mu}_F(z),[0.5,0.5]\}:z\in x+y\}$ $<\inf\{\tilde{\mu}_F(z):z\in x+y\}.$ It follows that for every $z\in x+y,[0.5,0.5]<\tilde{t}\le\max\{\tilde{\mu}_F(z),[0.5,0.5]\},$ and so $\tilde{t}\le\tilde{\mu}_F(z).$ This implies that $z\in U(F;\tilde{t}).$ Hence $x+y\in U(F;\tilde{t}).$ Now, let $x,a\in U(F;\tilde{t}).$ Then by (ii), there exists $y\in R$ such that $x\in a+y$ and $\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x)\}\le\max\{\tilde{\mu}_F(y),[0.5,0.5]\}.$ To prove $y\in U(F;\tilde{t}),$ we recall that $[0.5,0.5]<\tilde{t}\le\tilde{\mu}_F(x)\le\min\{\tilde{\mu}_F(a),\tilde{\mu}_F(x)\}\le$ $\max\{\tilde{\mu}_F(y),[0.5,0.5]\}.$

Hence, it follows that $[0.5,0.5]\le\tilde{\mu}_F(y),$ and so $y\in U(F;\tilde{t}).$ Therefore $U(F;\tilde{t})$ is an H_v-subring of R. Now, assume that $[0.5,0.5]<\tilde{t}\le[1,1]$ and $x\in U(F;\tilde{t})$ and $c\in R.$ Then $[0.5,0.5]<\tilde{t}\le\tilde{\mu}_F(x)\le\inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z),[0.5,0.5]\}\\ :z\in c\cdot x\end{matrix}\right\}.$ It follows that for every $z\in c\cdot x,[0.5,0.5]<\tilde{t}\le\max\{\tilde{\mu}_F(z),[0.5,0.5]\},$ and so $\tilde{t}\le\tilde{\mu}_F(z).$ This implies $z\in U(F;\tilde{t}),$ and hence, $c\cdot x\in U(F;\tilde{t}).$

In [24], Yuan et al. introduced the concept of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and also the fuzzy subgroup proposed by Bhakat and Das. Based on [24], we can extend the fuzzy subgroup with thresholds to the interval valued fuzzy H_v-ideals with thresholds expressed in the following way:

Definition 4.8. Let $s,t\in[0,1]$ and $\tilde{s}<\tilde{t}.$ Then an interval valued fuzzy set F of R is called an interval valued fuzzy H_v-ideal with thresholds $(\tilde{s},\tilde{t})$ of R if the following conditions hold:

$(I)\min\{\tilde{\mu}_F(x),\tilde{\mu}_F(y),\tilde{t}\}\le\inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z),\tilde{s}\}\\ :z\in x+y\end{matrix}\right\},$ for all $x,y\in R,$

(II) For all $x, a \in R,$ there exists $y \in R$ such that $x \in a + y$ and

$\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), \tilde{t}\} \leq \max\{\tilde{\mu}_F(y), \tilde{s}\},$

(III) For all $x, a \in R,$ there exists $z \in R$ such that $x \in z + a$ and

$\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), \tilde{t}\} \leq \max\{\tilde{\mu}_F(z), \tilde{s}\},$

$(IV) \min\{\tilde{\mu}_F(y), \tilde{t}\} \leq \inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z), \tilde{s}\}: \\ z \in x \cdot y\end{matrix}\right\}$

for all $x \in R.$

If F is an interval valued fuzzy H_v-ideal with thresholds of R, then we conclude that F is an ordinary interval valued fuzzy H_v-ideal when $\tilde{s} = [0,0], \tilde{t} = [1,1];$ and F is an interval valued $(\in, \in \vee q)$-fuzzy H_v-ideal when $\tilde{s} = [0,0], \tilde{t} = [0.5, 0.5].$

Now, we characterize the interval valued fuzzy H_v-ideal with thresholds by using their level H_v-ideals.

Theorem 4.9. An interval valued fuzzy set F of R is an interval valued fuzzy H_v-ideal with thresholds $(\tilde{s}, \tilde{t})$ of R if and only if $U(F; \tilde{\alpha})(\neq \phi)$ is an H_v-ideal of R for all $\tilde{s} < \tilde{\alpha} \leq \tilde{t}.$

Proof. Let F be an interval valued fuzzy H_v-ideal with thresholds $(\tilde{s}, \tilde{t})$ of R and $\tilde{s} < \tilde{\alpha} \leq \tilde{t}.$ Let $x, y \in U(F; \tilde{\alpha}).$ Then $\tilde{\mu}_F(x) \geq \tilde{\alpha}$ and $\tilde{\mu}_F(y) \geq \tilde{\alpha}.$ Now, we have $\inf\{\max\{\tilde{\mu}_F(z), \tilde{s}\}: z \in x + y\} \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), \tilde{t}\} \geq$ $\max\{\tilde{\alpha}, \tilde{t}\} \geq \tilde{\alpha} > \tilde{s}.$

Hence, for every $z \in x + y,$ we have $\max\{\tilde{\mu}_F(z), \tilde{s}\} > \tilde{\alpha} > \tilde{s}.$ This implies that $\tilde{\mu}_F(z) \geq \tilde{\alpha},$ and hence $z \in U(F; \tilde{\alpha}).$ Consequently, we obtain $x + y \subseteq U(F; \tilde{\alpha}).$ Now, let $x, a \subseteq U(F; \tilde{\alpha}),$ then there exists $y \in R$ such that $x \in a + y$ and $\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), \tilde{t}\} \leq \max\{\tilde{\mu}_F(y), \tilde{s}\}.$

From $x, a \subseteq U(F; \tilde{\alpha}),$ we have $\tilde{s} < \tilde{\alpha} \leq \min\{\tilde{\alpha}, \tilde{t}\} \leq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), \tilde{t}\}$ $\leq \max\{\tilde{\mu}_F(y), \tilde{s}\}.$

This implies that $\tilde{\mu}_F(y) \geq \tilde{\alpha},$ hence, $y \in U(F; \tilde{\alpha}).$ Therefore, $U(F; \tilde{\alpha}). = a + U(F; \tilde{\alpha}),$ for all $a \in U(F; \tilde{\alpha}).$ Similarly,we can get $U(F; \tilde{\alpha}) + a = U(F; \tilde{\alpha}),$ for all $a \in U(F; \tilde{\alpha}).$

Now, let $x \in U(F; \tilde{\alpha})$ Then, $\tilde{\mu}_F(x) \geq \tilde{\alpha}.$ Consequently, we have $\inf\{\max\{\tilde{\mu}_F(z), \tilde{s}\}: z \in c \cdot x\} \geq \min\{\tilde{\mu}_F(x), \tilde{t}\} \geq$ $\min\{\tilde{\alpha}, \tilde{t}\} \geq \tilde{\alpha} > \tilde{s},$

and so, for every $z \in c \cdot x,$ we have $\max\{\tilde{\mu}_F(z), \tilde{s}\} > \tilde{\alpha} > \tilde{s}.$ This implies that $\tilde{\mu}_F(z) \geq \tilde{\alpha},$ and thereby, $z \in U(F; \tilde{\alpha}).$ Hence, $c \cdot x \subseteq U(F; \tilde{\alpha}).$ Thus, $U(F; \tilde{\alpha})$ is an H_v-ideal of R, for all $\tilde{s} < \tilde{\alpha} \leq \tilde{t}.$

Conversely, let F be an interval valued fuzzy set of R such that $U(F; \tilde{\alpha})(\neq \phi)$ is an H_v-ideal of R for all $\tilde{s} < \tilde{\alpha} \leq \tilde{t}.$ If there exist $x, y, z \in R$ with $z \in x + y$ such that $\max\{\tilde{\mu}_F(z), \tilde{s}\} < \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), \tilde{t}\} = \tilde{\alpha},$ then $\tilde{s} < \tilde{\alpha} \leq \tilde{t}, \tilde{\mu}_F(z) < \tilde{\alpha}$ and $x, y \in U(F; \tilde{\alpha}).$ Since $U(F; \tilde{\alpha})$ is an H_v-ideal of R and $x, y \in U(F; \tilde{\alpha}), x + y \subseteq U(F; \tilde{\alpha}).$ Hence, $\tilde{\mu}_F(z) \geq \tilde{\alpha}$ for all $z \in x + y.$ However, this contradicts $\tilde{\mu}_F(z) < \tilde{\alpha}.$ Therefore, $\min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), \tilde{t}\} \leq \max\{\tilde{\mu}_F(z), \tilde{s}\},$ for all $x, y, z \in R$ with $z \in x + y.$ This implies that

$\min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), \tilde{t}\} \leq \inf\left\{\begin{matrix}\max\{\tilde{\mu}_F(z), \tilde{s}\}: \\ z \in x + y\end{matrix}\right\},$

for all $x, y \in R.$ This proves that the condition (I) of Definition 4.8 is held.

Now, we assume that there exist $x_0, a_0 \in R$ such that for all $y \in R$ satisfies $x_0 \in a_0 + y,$ the following inequality is held:

$$\max\{\tilde{\mu}_F(y), \tilde{s}\} < \min\{\tilde{\mu}_F(a_0), \tilde{\mu}_F(x_0), \tilde{t}\} = \tilde{\alpha}.$$

Then $\tilde{s} < \tilde{\alpha} \le \tilde{t}, x_0, a_0 \in U(F;\tilde{\alpha})$ and $\tilde{\mu}_F(z) < \tilde{\alpha}.$ Since $x_0, a_0 \in U(F;\tilde{\alpha})$ and $U(F;\tilde{\alpha})$ is an H_v-ideal, there exists $y_0 \in U(F;\tilde{\alpha})$ such that $x_0 \in a_0 + y.$ From $y_0 \in U(F;\tilde{\alpha}),$ we get $\tilde{\mu}_F(y_0) \ge \tilde{\alpha}.$ This is contradicts $\tilde{\mu}_F(y_0) < \tilde{\alpha}.$ Therefore, $\min\{\tilde{\mu}_F(a), \tilde{\mu}_F(x), \tilde{t}\} \le \max\{\tilde{\mu}_F(y), \tilde{s}\}.$

Hence, the condition (II) of Definition 4.8 is held. Similarly, we can also prove that condition (III) of Definition 4.8 is held.

If there exist $x, z \in R$ with $z \in c \cdot x$ such that $\max\{\tilde{\mu}_F(z), \tilde{s}\} < \min\{\tilde{\mu}_F(x), \tilde{t}\} = \tilde{\alpha},$ then $\tilde{s} < \tilde{\alpha} \le \tilde{t}, \tilde{\mu}_F(z) < \tilde{\alpha}$ and $x \in U(F;\tilde{\alpha}).$ Since $U(F;\tilde{\alpha})$ is an H_v-ideal of R and $x \in U(F;\tilde{\alpha}), c \cdot x \subseteq U(F;\tilde{\alpha}).$ Hence $\tilde{\mu}_F(z) \ge \tilde{\alpha}$ for all $z \in c \cdot x.$ This clearly contradicts to $\tilde{\mu}_F(z) < \tilde{\alpha}.$ Therefore, $\min\{\tilde{\mu}_F(x), \tilde{t}\} \le \max\{\tilde{\mu}_F(z), \tilde{s}\}$ for all $x, z \in R$ with $z \in c \cdot x.$ This implies that $\min\{\tilde{\mu}_F(x), \tilde{t}\} \le \inf\{\max\{\tilde{\mu}_F(z), \tilde{s}\} : z \in c \cdot x\}$ for all $x \in R.$ Hence, condition (IV) of Definition 4.8 holds. Thus, we have proved that F is an interval valued fuzzy H_v-ideal with thresholds $(\tilde{s}, \tilde{t})$ of R.

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