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GLOBAL EXISTENCE IN REACTION DIFFUSION NONLINEAR PARABOLIC PARTAIL DIFFERENTIAL EQUATION IN IMAGE PROCESSING

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Abstract

This paper deals with the equation.

$$(P)\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[g(|\nabla u_\sigma|)\nabla u] = f(t,x,u,\nabla u) \text{ in } Q_T \\ u(0,x) = u_0(x) \geq 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma_T, \end{cases}$$

where $\Omega =]0,1[\times]0,1[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$, where $(T > 0)$, $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-(\frac{|x|^2}{4\sigma})}$, $x \in \mathbb{R}, \sigma > 0$ and $\nabla u_\sigma = \nabla(u * G_\sigma) = u * \nabla G_\sigma$.

We give the proof of global existence of our nonlinear reaction diffusion of problem (P). In the first we define an approximating scheme and by using Schauder fixed point theorem in ordered Banach spaces, we show the existence of a solution for this approached problem. Finally by making some estimations we prove that the solution of the truncated equation converge to the solution of our problem. We use a new technique recently introduced in order to generalize some of interesting prior works that has been presented.

Keywords: Weak solutions, Non linear restoration diffusion, Image processing.

Introduction

In recent years attention has been given to weak solutions of elliptic problems under linear boundary conditions, and different methods for the existence problem have been used [1], [4], [5], [12] and [13]. The corresponding parabolic case equations have also been studied by many authors, see for instance [1], [14] and [17].

Moreover, image processing is always a challenging problem, this topic has become "hot", recently and a very active field of computer applications and research [18]. Image is restored to its original quality by inverting the physical degradation phenomenon such as defocus, linear motion, atmospheric degradation and additive noise. Partial differential equation (PDE) methods in image processing have proven to be fundamental tools for image diffusion and restoration [7], [8], [9] and [21].

In this paper, we prove the existence of solutions for the reaction- diffusion of the form:

$$(P)\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}[g(|\nabla u_\sigma|)\nabla u] = f(t,x,u,\nabla u) \text{ in } Q_T \\ u(0,x) = u_0(x) \geq 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma_T, \end{cases}$$

where $\Omega =]0,1[\times]0,1[$, $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$, where $(T > 0)$, $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-(\frac{|x|^2}{4\sigma})}$, $x \in \mathbb{R}, \sigma > 0$ and $\nabla u_\sigma = \nabla(u * G_\sigma) = u * \nabla G_\sigma$.
In this study, we need the following assumptions :
(H_g) $g : [0,+\infty[\to [0,+\infty[$ is a smooth non-increasing function, where $g(0) > 0$ and $\lim_{s\to+\infty} g(s) = 0$.
$(H_f)_1$ $f : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable and $f(t,x,.,.) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are locally Lipshitz continuous: $(\exists r > 0$, for almost $(t,x) \in Q_T/$ $|f(t,x,u,p) - f(t,x,\hat{u},\hat{p})| \leq k(r)[|u-\hat{u}| + \|p-\hat{p}\|])$ for all $0 \leq |u|, |\hat{u}|, \|p\|, \|\hat{p}\| \leq r$.
$(H_f)_2$ for almost $(t,x) \in Q_T$, $f(t,x,0,0) \geq 0$.
$(H_f)_3$ $\forall(u,p) \in \mathbb{R}\times\mathbb{R}^N$ and for almost $(t,x) \in Q_T$, $uf(t,x,u,p) \leq 0$.
$(H_f)_4$ $|f(t,x,u,\nabla u)| \leq C(|u|)[F(t,x) + |\nabla u|^2]$ where $C : [0,+\infty[\to [0,+\infty[$ is non-decreasing, $F \in L^1(Q_T)$ and $|\nabla u|^2 = (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2$.

In the present paper: We give the proof of global existence of our nonlinear reaction diffusion of problem (P), this is done in four steps: the first step we show the positive solution. In the second step is to truncate the equation and shows that the problem obtained has a solution. In the third step we establish appropriate estimates on the approximate solutions. In the last step, we show the convergence of the approximate equation. We use a new technique recently introduced, in fact our results $f = f(t, x, u, \nabla u)$ are a generalization of the work $f = 0$ presented by Catté [16], and the work $f = f(t, x, u)$ presented by Alaa [2]. Now we will recall some functional spaces that will be used throughout this paper. $L^2(0, T, H^1(\Omega))$ is the set of functions u such that, for all every $t \in (0, T)$, $u(t)$ belongs to $H^1(\Omega)$ with the norm

$$\|u\|_{L^2(0,T,H^1(\Omega))} = \left(\int_0^T \|u(t)\|^2_{H^1(\Omega)} dt\right)^{\frac{1}{2}}.$$

$L^\infty(0,T;C^\infty(\Omega))$ is the set of functions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $C^\infty(\Omega)$ with the norm

$$\|u\|_{L^\infty(0,T,C^\infty(\Omega))} = \inf\{c, \|u(t)\|_{C^\infty(\Omega)} \leq c \text{ on } (0,T)\}.$$

THE MAIN RESULT

A function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is a weak solution of (P) if for all $\varphi \in C^1(Q_T)$ such that $\varphi(T,.) = 0$,

$$\int_{Q_T} [-u\frac{\partial\varphi}{\partial t} + g(|\nabla u_\sigma|)\nabla u \nabla\varphi] dtdx = \int_{Q_T} f(t,x,u(t,x),\nabla u(t,x))\varphi dtdx + \int_\Omega u_0\varphi(0,x)dx.$$

Our main result in this paper is the following existence theorem. Assume that $(H_f)_1$, $(H_f)_2$, $(H_f)_3$, $(H_f)_4$, and (H_g), and that for all

$$R \geq 0, \sup_{|u|\leq R}(|f(t,x,u,\nabla u)|) \in L^1(Q_T).$$

Then
(i) Problem (P) admits a weak positive solution.
(ii) If moreover for all $r \geq 1, f(t,x,r,p) \leq 0$ and $u_0(x) \leq 1$, we have $0 \leq u(t,x) \leq$ 1 dans Q_T.
A typical examples where the result of this paper can be applied are (i) thre of the diffusivity Perana and Malik [21] are

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$g(s) = \frac{1}{1+(\frac{s}{\lambda})^2}$ or $g(s) = exp(-(1+\frac{s}{\lambda}))$
where $\lambda > 0$. Or

$$g(s) = \frac{d}{\sqrt{1+\eta(\frac{s}{\lambda})^2}}$$

where $\eta \geq 0$, $d > 0$ and λ is a threshold (contrast) parameter.
(ii) $f(t,x,u,\nabla u) = -\beta u(u-a)^{2\alpha}(1-u)^{2\gamma} + a_{11}u|\nabla u|^{\alpha}, 1 \leq \alpha < 2,$ where $\beta, \gamma > 0, \alpha, \gamma \neq \frac{1}{2}$ and $0 < a < 1$, $a_{11} \leq 0$.

Proof of theorem:
The proof of (i) is done in four steps
Step (1): Positivity of the solutions
Consider the function

$$sign^{-}(s) = \begin{cases} -1 \text{ if } s < 0 \\ 0 \text{ if } s \geq 0. \end{cases}$$

Let $\varepsilon > 0$ we build a sequence of regular convex functions $\jmath_{\varepsilon}(s)$
such as $\jmath_{\varepsilon}^{'}(s)$ is bounded and for all $s \in \mathbb{R}$, $\jmath_{\varepsilon}(s) \to s^{-}$, $\jmath_{\varepsilon}^{'}(s) \to sign^{-}(s)$ when $\varepsilon \to 0$. (a typical example of $\jmath_{\varepsilon}(s)$ can be given by

$$\jmath_{\varepsilon}(s) = \begin{cases} -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\exp(-\varepsilon \int_0^s \frac{t}{t-\varepsilon}dt) \text{ if } s < 0 \\ 0 \text{ if } s \geq 0,) \end{cases}$$

Let u be a solution of (1), we multiply both sides of the first equation of (P) by $\jmath_{\varepsilon}^{'}(u)$ and by integrating on $Q_t =]0,t[\times\Omega$ for $t \in [0,T[$, we obtain

$$\int_{Q_t} j_\varepsilon'(u)\frac{\partial u}{\partial t}dxdt + \int_{Q_t} A\nabla u.\nabla j_\varepsilon'(u)dxdt = \int_{Q_t} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds$$

where $A(t,x) = g(|\nabla u_\sigma|) \in L^\infty(0,T;C^\infty(\Omega))$ because $u \in L^\infty(0,T;L^2(\Omega))$ and g, G_σ are C^∞ such as

$$\|\nabla u_\sigma\|_{L^\infty(Q_T)} \leq C_0.$$

Moreover as (H_g) g is non-increasing, then there $a = g(C_0) > 0$ which depends only on σ and on $\|u_0\|_{L^2(\Omega)}$ such as:

$$A(t,x) \geq a, \forall (t,x) \in Q_T.$$

Consequently,

$$\int_\Omega [j_\varepsilon(u)(t) - j_\varepsilon(u)(0)]dx + a\int_{Q_t} |\nabla u|^2 j_\varepsilon''(u)dsdx \leq \int_{Q_t} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds$$

Since $u(0,x) = u_0 \geq 0 \Rightarrow j_\varepsilon(u_0(x)) = j_\varepsilon(u)(0,x) = 0 \Rightarrow \int_\Omega j_\varepsilon(u)(0)dx = 0$ and $\int_{Q_t} |\nabla u|^2 j_\varepsilon''(u)dsdx \geq 0$. then we have

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{Q_t} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds$$

$$\leq \int_{(0,t)\times[u<0]} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds + \int_{(0,t)\times[u\geq 0]} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds$$

where $u \geq 0$, we have $j_\varepsilon'(u) = 0$ and $\int_{(0,t)\times[u\geq 0]} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds = 0$, therefore

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{(0,t)\times[u<0]} f(s,x,u,\nabla u)j_\varepsilon'(u)dxds.$$

When $\varepsilon \to 0$, we obtain

$$\int_\Omega (u)^-(t)dx \leq -\int_{(0,t)\times[u<0]} f(s,x,u,\nabla u)dxds \leq 0 \text{ Using } (H_f)_3$$

and the fact that $(u)^-(t) \geq 0$, we obtain $(u)^-(t) = 0$ on Ω
therefore

$$u \geq 0 \text{ in } Q_T.$$

Step (2): An existence result when f is bounded

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Assume $(H_f)_2, (H_f)_3$, and that there exists $M > 0$, such as for almost $(t,x) \in Q_T$ and $\forall r \in \mathbb{R}$,

$$|f(t,x,r,p)| \leq M.$$

Then for all $u_0 \in L^2(\Omega)$, problem (P) admits a weak solution. Moreover, there exists $C = C(M, a, T, \|u_0\|_{L^2(\Omega)})$ such that

$$\sup_{0<t<T} \|u(t)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Proof

We will show the existence of a weak solution by the classical Schauder fixed point theorem. Firstly we introduce the space

$$W(0,T) = \{v \in L^2(0,T;H^1(\Omega)) : \frac{dv}{dt} \in L^2(0,T;(H^1(\Omega))^{'})\}$$

which is a Hilbert space for the graph norm. Let $v \in W(0,T) \cap L^\infty(0,T;L^2(\Omega))$ and we consider $u(v)$ the solution of the linear problem

$$u(v) \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

$$\text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0$$

$$\int_{Q_T}[-u(v)\frac{\partial \varphi}{\partial t} + g(|\nabla v_\sigma|)\nabla u(v)\nabla\varphi]dtdx = \int_{Q_T} f(t,x,v(t),\nabla v)\varphi dtdx + \int_\Omega u_0\varphi(0,x)dx$$

We take $\varphi = u(v)$ in (9), and deduce that for all $0 < t < T$,

$$\frac{1}{2}\int_\Omega u(v)^2(t) + \int_{Q_t} g(|\nabla v_\sigma|)|\nabla u(v)|^2 = \int_{Q_t} f(t,x,v(t),\nabla v)u(v) + \frac{1}{2}\int_\Omega u_0^2 dx$$

Using (3) and the assumption (6) on f,

$$\int_{Q_t} f(s,x,u(s),\nabla v(s))u(v)dxds \leq \left(\int_{Q_t} f^2\right)^{\frac{1}{2}} \left(\int_{Q_t} (u(v))^2\right)^{\frac{1}{2}}$$

$$\leq M|Q_t|^{\frac{1}{2}} \left(\int_{Q_t} (u(v))^2\right)^{\frac{1}{2}}$$

$$\leq M\left[[\sqrt{\varepsilon}|Q_t|]^{\frac{1}{2}}[\frac{1}{\sqrt{\varepsilon}}\int_{Q_t} u(v)^2]^{\frac{1}{2}}\right]$$

$$\leq M\left[\frac{1}{2}\varepsilon|Q_t| + \frac{1}{2}\frac{1}{\varepsilon}\int_{Q_t} u(v)^2\right]$$

we set

$$\frac{1}{2}\varepsilon|Q_t| = 1 \Rightarrow \frac{1}{\varepsilon} = \frac{|Q_t|}{2}$$

$$\int_{Q_t} f(s,x,u(s),\nabla v(s))u(v) \leq M\left[1 + \frac{|Q_t|}{4}\int_{Q_t} u(v)^2\right]$$

$M' = sup(1 + \frac{|Q_t|}{4})$, then

$$\int_{Q_t} f(s,x,u(s),\nabla v(s))u(v) \leq M(1 + \int_{Q_t} u(v)^2) \text{ where } MM' \text{ denote } M$$

Using (3), we obtain

$$\frac{1}{2}\int_\Omega u(v)^2(t) + a\int_{Q_t} |\nabla u(v)|^2 \leq M(1 + \int_{Q_t} u(v)^2) + \frac{1}{2}\int_\Omega u_0^2 dx$$

Now by Gronwall's lemma, we obtain the estimation (7). In fact
we set $y(t) = \int_\Omega u(v)^2(t)dx$

$$y(t) - 2M\int_0^t y(s)ds \leq y(t) - 2M\int_0^t y(s)ds + 2a\int_{Q_t} |\nabla u(v)|^2 \leq 2M + \|u_0\|^2_{L^2(\Omega)}$$

where

$$\frac{d}{dt}[\int_0^t y(s)dse^{-2Mt}] = [y(t) - 2M\int_0^t y(s)ds]e^{-2Mt}$$

implies

$$[y(t) - 2M\int_0^t y(s)ds]e^{-2Mt} \leq e^{-2Mt}[2M + \|u_0\|^2_{L^2(\Omega)}]$$

$$\int_0^t y(s)dse^{-2Mt} \leq (2M + \|u_0\|^2_{L^2(\Omega)})[-\frac{1}{2M}e^{-2Ms}]_{s=0}^{s=t}$$

$$\int_0^t y(s)dse^{-2Mt} \leq [1 + \frac{1}{2M}\|u_0\|^2_{L^2(\Omega)}][1 - e^{-2Mt}], t \leq T$$

so $u \in L^2(0,T,H^1(\Omega))$

$$\int_0^t y(s)ds \leq [1 + \frac{1}{2M}\|u_0\|^2_{L^2(\Omega)}][e^{2Mt} - 1] \leq C(T)$$

These estimates lead us to introduce the space

$$W_0(0,T) = \{v \in W(0,T) \cap L^\infty(0,T;L^2(\Omega)) : v(0) = u_0 \text{ and}$$
$$\sup_{0<t<T} \|u(t)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H^1(\Omega))} \leq C\},$$

where $C = C(M, a, T, \|u_0\|_{L^2(\Omega)})$ is the constant obtained in (7). Then we define the application :

$$F : W_0(0,T) \to W_0(0,T)$$

$$v \mapsto F(v) = u(v), \text{ where } u \text{ is a solution of (9).}$$

show that $F(W_0(0,T)) \subset W_0(0,T)$, let $v \in W_0(0,T) \Rightarrow v \in W(0,T)$ then $F(v) = u \Rightarrow u$ verified, so $F(W_0(0,T)) \subset W_0(0,T)$.
$W_0(0,T)$ is a nonempty closed convex in $W(0,T)$, and it injects $W_0(0,T) \circlearrowleft L^2(0,T; L^2(\Omega))$ is compact. To apply the Schauder fixed point theorem, we show that F is weakly continuous from $W_0(0,T)$ in $W_0(0,T)$. Then consider a sequence (v_n) in $W_0(0,T)$ such as $v_n \rightharpoonup v$ in $W_0(0,T)$ and let $u_n = F(v_n)$. According to the classical results of compactness, we can extract from the sequence (u_n) a subsequence yet denoted (u_n) such that

· $u_n \rightharpoonup u$ weakly in $L^2(0,T; H^1(\Omega))$
· $u_n \rightharpoonup u$ strongly in $L^2(0,T; L^2(\Omega))$ and almost everywhere in Q_T.
· $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(0,T; L^2(\Omega))$.
· $v_n \rightharpoonup v$ strongly in $L^2(0,T; L^2(\Omega))$ and almost everywhere in Q_T.
· $\nabla G_\sigma * v_n \rightharpoonup \nabla G_\sigma * v$ strongly in $L^2(0,T; L^2(\Omega))$ and almost everywhere in Q_T.
· $g(|\nabla G_\sigma * v_n|) \rightharpoonup g(|\nabla G_\sigma * v|)$ strongly in $L^2(0,T; L^2(\Omega))$.
· $f(t,x,v_n,\nabla v_n) \rightharpoonup f(t,x,v,\nabla v)$ strongly in $L^1(Q_T)$.

The latter is obtained by applying the dominated convergence theorem. We can then pass to the limit in (9), with v_n instead of v, and obtain that $u = F(v)$. By uniqueness of the solution of (9), then the sequence $u_n = F(v_n)$ converges weakly to $u = F(u)$ in $W_0(0,T)$. We deduce the existence of $u \in W_0(0,T)$ such as $u = F(u)$, and thus the existence of $u \in W(0,T)$.

Step (3): Approximate problem and a priori estimates

Consider the truuncation function $\psi_n \in C_0^\infty(\mathbb{R})$ defined by

$$\psi_n(r) \begin{cases} 1 \text{ if } |r| \le n, \\ 0 \text{ if } |r| \ge n+1. \end{cases}$$

We truncate the nonlinearity f by ψ_n,

$$f_n(t,x,u,p) = [\psi_n(|u| + \|p\|) f(t,x,u,p)] * \rho_n(t,x)$$

where $\rho_n \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$, $supp\rho_n \subset B(0, \frac{1}{n})$ $\int \rho_n = 1, \rho_n \ge 0$ on $(\mathbb{R} \times \mathbb{R}^N)$ $\rho_n(y) = n^N \rho(ny)$. We also consider non-decreasing sequences $u_0^n \in C_0^\infty(\Omega)$ such that $u_0^n \to u_0$ in $L^2(\Omega)$ and

$$|f_n(t,x,r,p)| \le M_n$$

estimate (6) is applied, then we can deduce the existence of a weak solution of the problem:

$$(P_1) \begin{cases} \frac{\partial u_n}{\partial t} - \text{div}[g(|\nabla (u_n)_\sigma|)\nabla u_n] = f_n(t,x,u_n,\nabla u_n) \text{ in } Q_T \\ u_n(0,x) = u_0^n(x) \ge 0 \text{ on } \Omega \\ \frac{\partial u_n}{\partial \nu} = 0 \text{ on } \Sigma_T, \end{cases}$$

Since $u_0^n \geq 0$ on Ω, the $(H_f)_2$ assures that $u_n \geq 0$ is in Q_T. Moreover, under the assumption $(H_f)_3$ we have also $u_n f_n(t,x,u_n,\nabla u_n) \leq 0$ in Q_T.

Now we will show that a subsequence u_n converges to the weak solution u of problem (P). For this we need to prove the following result:

Let (u_n) the sequence of weak solutions defined by (P_1), then we have :
(i) $\int_{Q_T} |f_n(t,x,u_n,\nabla u_n)|dxdt \leq \int_\Omega |u_0^n|dx$,
(ii) u_n is bounded in $L^2(0,T;H^1(\Omega))$ et

$$\int_{Q_T} |u_n f_n(t,x,u_n,\nabla u_n)|dxdt \leq \frac{1}{2}\int_\Omega (u_0^n)^2 dx,$$

(iii) u_n is relatively compact in $L^2(Q_T)$.

Proof
(i) By remark (0.2.2) $|f_n(t,x,u_n,\nabla u_n)| = -f_n(t,x,u_n,\nabla u_n)$. Thus by integrating the equation satisfied by u_n in Q_T we obtain

$$\int_\Omega u_n(T)dx - \int_{Q_T} f_n(t,x,u_n,\nabla u_n) = \int_\Omega u_0^n dx,$$

therefore

$$\int_{Q_T} |f_n(t,x,u_n,\nabla u_n)|dxdt \leq \int_\Omega |u_0^n|dx$$

(ii) Firstly we show that u_n is bounded in $L^2(Q_T)$, for this we consider $\varphi = u_n$ as a function test in (P_1), we then deduce that

$$\frac{1}{2}\int_\Omega u_n^2(t) + \int_{Q_t} g(|\nabla(u_n)_\sigma|)|\nabla u_n|^2 = \int_{Q_t} f(t,x,u_n,\nabla u_n)u_n + \frac{1}{2}\int_\Omega (u_0^n)^2 dx$$

Then we use (3) and the hypothesis (4) on f to obtain

$$\frac{1}{2}\int_\Omega u_n^2(t) + a\int_{Q_t} |\nabla u_n|^2 \leq \frac{1}{2}\int_\Omega (u_0^n)^2 dx.$$

We have also

$$\int_{Q_T} |u_n f_n(t,x,u_n,\nabla u_n)|dxdt \leq \frac{1}{2}\int_\Omega (u_0^n)^2 dx,$$

where we have

$$\sup_{0<t<T} \|u_n(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)},$$

$$\|u_n\|_{L^2(0,T;H^1(\Omega))} \leq (1+\frac{1}{2a})\|u_0\|_{L^2(\Omega)}.$$

(iii) Since $\frac{\partial u_n}{\partial t} =$ div $(A_n \nabla u_n) + f_n(t,x,u_n,\nabla u_n)$ is bounded in $L^1(0,T;(H^1(\Omega))') + L^1(\Omega)$. Since u_n is also bounded $L^2(0,T;H^1(\Omega))$ and that the injection $H^1(\Omega) \circlearrowleft L^2(\Omega)$ is compact, it follows that u_n is relatively compact in $L^2(Q_T)$.

We set $T_k(s) = max\{-k, min(k,s)\}$ and $G_k(s) = s - T_k(s)$ we remark that for $0 \leq s \leq k$, $T_k(s) = s$ and $T_k(s) = k$ for $s > k$.

There exists a constant R_2 depending on k and $\|u_0\|_{L^1(\Omega)}$, such that

$$\int_{Q_T} |\nabla T_k(u_n)|^2 \leq R_2$$

Proof

We multiply the first equation (P_1) by $T_k(u_n)$ and we integrate on Q_T, we obtain

$$\int_{Q_T} T_k(u_n)\frac{\partial u_n}{\partial t} - \int_{Q_T} T_k(u_n)\text{div}(g(|\nabla(u_n)_\sigma|)\nabla u_n) = \int_{Q_T} T_k(u_n) f_n(t,x,u_n,\nabla u_n)$$

$$\int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n) + a\int_{Q_T} |\nabla T_k(u_n)|^2 \leq k\int_{Q_T} |f_n|.$$

We set $S_k(r) = \int_0^r T_k(s)ds$, $S_k(u_n) = \int_0^{u_n} T_k(s)ds$

$$\frac{dS_k(u_n)}{dt} = T_k(u_n)\frac{\partial u_n}{\partial t}$$

$$\int_0^T \frac{\partial u_n}{\partial t} T_k(u_n) = \int_0^T \frac{dS_k(u_n)}{dt} = S_k(u_n(T)) - S_k(u_n(0))$$

Then

$$\int_\Omega S_k(u_n(T)) + a\int_{Q_T} |\nabla T_k(u_n)|^2 \leq k\int_{Q_T} |f_n| + \int_\Omega S_k(u_n(0))$$

$S_k(u_n(T)) \geq 0$ and for all $r \geq 0$, $|S_k(r)| \leq \frac{k^2}{2} + k(r-k)^+$.
by using the result

$$\int_{Q_T} |f_n(t,x,u_n,\nabla u_n)|dxdt \leq cte = R_1$$

we have

$$a\int_{Q_T} |\nabla T_k(u_n)|^2 \leq k\int_{Q_T} |f_n| + \int_\Omega \frac{k^2}{2} + k(u_n(0)-k)^+$$

$$a\int_{Q_T}|\nabla T_k(u_n)|^2 \le kR_1 + \frac{k^2}{2}|\Omega| + k\int_{\Omega}(u_n(0)-k)^+ \le C(k,\|u_n(0)\|)$$

$$\int_{Q_T}|\nabla T_k(u_n)|^2 \le C(k,\|u_n(0)\|) = R_2.$$

Step (4): Convergence

Our objective is to show that $u_n \to u$ solution of the problem (P_1). The seqquence u_0^n is uniformly bounded in $L^1(\Omega)$ (since it converge in $L^2(\Omega)$) and by $\int_{Q_T}|f_n(t,x,u_n,\nabla u_n)| \le R_1$ the non-linearitie f_n is uniformly bounded in $L^1(\Omega)$. Then according to a result of Barras, Hassan and L.Veron [10] the application

$$L^1(\Omega)\times L^1(Q_T) \mapsto L^1(0,T;W^{1,1}(\Omega)) \text{ is compact}$$

$$(u_0^n, f_n) \mapsto u_n$$

Therefore, we can extract a subsequence, still denoted by u_n such that

$$u_n \to u \text{ in } L^1(0,T;W^{1,1}(\Omega))$$

$$u_n \to u \text{ almost everywhere in } Q_T$$

$$\nabla u_n \to \nabla u \text{ almost everywhere in } Q_T.$$

Since f_n is continuou, we have

$$f_n(t,x,u_n,\nabla u_n) \to f(t,x,u,\nabla u) \text{ almost everywhere in } Q_T.$$

This is not sufficient to ensure that u is a solution of (P_1). In fact, we have to prove that the previous convergence is in $L^1(Q_T)$. In view of the Vitali theorem, to show that

$$f_n(t,x,u_n,\nabla u_n) \to f(t,x,u,\nabla u) \text{ in } L^1(Q_T).$$

Is equivalent to proving that $f_n(t,x,u_n,\nabla u_n)$ is equi-integrable in $L^1(Q_T)$.

$f_n(t,x,u_n,\nabla u_n)$ is equi-integrable in $L^1(Q_T)$.

The proof of this lemma requires the following result based on some properties of two time-regularization denoted by u_γ and u_σ $(\gamma,\sigma > 0)$, if $u \in L^2(0,T,H_0^1(\Omega))$ such that $u(0)=u_0 \in L^2(\Omega)$ we will denote by $w(s)$ a quantity such that $w(s) \to 0$ when $\varepsilon \to 0$ and $w^\sigma(\varepsilon)$ a quantity such that $w^\sigma(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Let $(u_n)_n$ be a sequence in $L^2(0,T;H_0^1(\Omega)) \cap C([0,T])$ such that $u_n(0)=u_0^n \in L^2(\Omega)$ and $(u_n)_t = \rho_{1,n}+\rho_{2,n}$ with $\rho_{1,n} \in L^2(0,T;H^{-1}(\Omega))$ and $\rho_{2,n} \in L^1(Q_T)$. Moreover assume that u_n converges to u in $L^2(Q_T)$, and u_0^n converges to $u(0)$ in $L^2(\Omega)$.
Let ψ be a function in $C^1([0,T])$ such that $\psi \ge 0$, $\psi' \le 0, \psi(T)=0$. Let φ be a Lipschitz increasing function in $C^0(\mathbb{R})$ such that $\varphi(0)=0$. Then, for all $k,\gamma > 0$,

$$\langle \rho_{1,n}, \psi\varphi(T_k(u_n)-T_k(u_m)_\gamma)\rangle + \int_{Q_T}\rho_{2,n}\psi\varphi(T_k(u_n)-T_k(u_m)_\gamma)$$

$$\geq w^{\gamma,n}(\frac{1}{m}) + w^{\gamma}(\frac{1}{n}) + \int_{\Omega} \psi(0)\phi(T_k(u) - T_k(u)_{\gamma})(0)dx$$
$$- \int_{\Omega} (u)(0)\psi(0)\varphi(T_k(u) - T_k(u)_{\gamma})(0),$$

where $\phi(t) = \int_0^t \varphi(s)ds$ and $G_k(s) = s - T_k(s)$.

Proof
(See [4], Lemma 7, p544). With $\rho_{1,n} = \operatorname{div}(A_n \nabla u_n) \in L^2(0,T,H^{-1}(\Omega))$, $\rho_{2,n} = f_n(t,x,u_n,\nabla u_n) \in L^1(Q_T)$.

Proof lemma 0.2.3
Let A be a measurable subset of Ω, for all $k \geq 0$, we have

$$\int_A |f_n(t,x,u_n,\nabla u_n)|dx = \int_{A\cap[u_n\leq k]} |f_n(t,x,u_n,\nabla u_n)|dx + \int_{A\cap[u_n>k]} |f_n(t,x,u_n,\nabla u_n)|dx$$

$$\frac{1}{k}\int_{A\cap[u_n>k]} k|f_n(t,x,u_n,\nabla u_n)|dx \leq \frac{1}{k}\int_{Q_T} u_n|f_n(t,x,u_n,\nabla u_n)|dx$$

$$\int_{A\cap[u_n>k]} |u_n f_n(t,x,u_n,\nabla u_n)|dx \leq \frac{1}{2k}\|u_0\|^2_{L^2(\Omega)}.$$

Now if we choose $k \geq \frac{\|u_0\|^2_{L^2(\Omega)}}{\varepsilon}$, then we have

$$\int_{A\cap[u_n>k]} |f_n(t,x,u_n,\nabla u_n)|dx \leq \frac{\varepsilon}{2}$$

$$\int_{A\cap[u_n>k]} |f_n(t,x,u_n,\nabla u_n)|dx \leq \frac{\varepsilon}{2} + \int_{A\cap[u_n\leq k]} |f_n(t,x,u_n,\nabla u_n)|dx$$
$$\leq \frac{\varepsilon}{2} + C_1(k)[\int_A F(t,x) + \int_{A\cap[u_n\leq k]} |\nabla u_n|^2]$$
$$\leq \frac{\varepsilon}{2} + C_1(k)[\int_A F(t,x) + \int_{A\cap[u_n\leq k]} |\nabla T_k(u_n)|^2]$$

[because $T_k(s) = s$ if $0 \leq s \leq k$], or

$$\int_{A\cap[u_n\leq k]} |\nabla T_k(u_n)|^2 = \int_{A\cap[u_n\leq k]} |[\nabla T_k(u_n) - \nabla T_k(u)] + \nabla T_k(u)|^2$$

$$\int_{A\cap[u_n\leq k]} |\nabla T_k(u_n)|^2 \leq 2\int_{A\cap[u_n\leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 + 2\int_A |\nabla T_k(u)|^2$$

we have

$$\int_{Q_T} |\nabla T_k(u_n)|^2 \leq C(k, \|u_0\|_{L^1(\Omega)})$$

so

$$|\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n \leq k]} \to 0 \text{ dans } L^1(\Omega) \text{ strongly}$$

$|\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n \leq k]}$ is equi-integrable in $L^1(\Omega)$
So, there exists $\rho_1 > 0$ such that if $|A| \leq \rho_1$ then

$$2C(k) \int_{A \cap [u_n \leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 < \frac{\varepsilon}{2}.$$

On the other hand, $F, |\nabla T_k(u)|^2 \in L^1(\Omega)$, therefore there exists $\rho_2 > 0$ such that $|A| \leq \rho_2$ we have

$$C_1(2 \int_A |\nabla T_k(u)|^2 + \int_A F(t,x)) < \frac{\varepsilon}{2}.$$

Choose $\rho_0 = \inf(\rho_1, \rho_2)$ if $|A| \leq \rho_0$ we obtain

$$\int_A |f_n(t,x,u_n,\nabla u_n)| dx \leq \varepsilon.$$

Using the following step (5), we complete the proof of Theorem (0.2.1).

Step (5) théorème (0.2.1)(ii) Let u be a weak solution of (1), and assume that $0 \leq u_0 \leq 1$ in Ω. Then $0 \leq u \leq 1$ in Q_T.

Proof
We have already obtained the positivity of weak solutions if the initial data is positive. So, we assume that $u_0 \leq 1$ and proof that $u \leq 1$. For this, we take $\overline{u} = 1 - u$, then we have $\nabla \overline{u} = -\nabla u$, we can verify that $\overline{u}$ satisfies

$$\overline{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1(\Omega)), \ f(t,x,1-\overline{u},\nabla\overline{u})$$
$$\text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0,$$

$$\int_{Q_T} -\overline{u}\frac{\partial \varphi}{\partial t} + g(|\nabla \overline{u}_\sigma|)\nabla\overline{u}\nabla\varphi = -\int_{Q_T} f(t,x,1-\overline{u},-\nabla\overline{u})\varphi + \int_\Omega \overline{u}_0 \varphi(0,x)dx$$

Then we consider the sequence of convex functions $j_\varepsilon'(r)$ is bounded and $\forall r \in \mathbb{R}$, $j_\varepsilon'(r) \to sign^-(r)$ when $\varepsilon \to 0$. We take $\varphi(s,x) = j_\varepsilon'(\overline{u})(s,x)\chi_{]0,t[}(s)$, for that $\varphi(t,x) = 0 \forall x \in \Omega$, as a test function in (21) and integrating with respect to $t \in]0,T[$ we obtain

$$\int_\Omega j_\varepsilon(\overline{u})(t)dx \leq \int_0^t \int_{[\overline{u}<0]} -f(s,x,1-\overline{u},\nabla\overline{u}) j_\varepsilon'(\overline{u}) ds dx$$

Passing to the limit as $\varepsilon \to 0^+$

$$\int_\Omega (\overline{u})^-(t)dx \le \int_0^t \int_{[u<0]} f(s,x,u,\nabla u)dsdx$$

using that $(H_f)_3$ we deduce

$$\int_\Omega (\overline{u})^-(t)dx \le 0.$$

Therefore $\overline{u}(t) \ge 0$ which implies $u = 1 - \overline{u} \le 1$.

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