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CONFIRMATION OF THE LEMOINE-LEVY CONJECTURE

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ABSTRACT

I confirm here the Lemoine-Levy conjecture, remained open since 1894, saying that: " $\forall n \geq 3 \exists (p, q)$ two prime integers such that 2n+1=p+2q". I use, essentially, some topological properties of the Integer part function as in my previous paper [4] published by the GJAETS in the June 2021 issue.

KEYWORDS: prime integer, prime-counting function, Lemoine-Levy conjecture, supremum, continuity 2010 Mathematics Subject Classification: A 11 xx (Number theory).

INTRODUCTION

Definition 1: We call "Le moine conjecture "or « Levy conjecture » (as it is the use in the Anglo-Saxons lands) or "the Lemoine-Levy conjecture" (as I prefer to call) the following assertion: " $\forall n$ an integer $\geq 3 \exists (p, q)$ two prime positive integers such that:$2n + 1 = p + 2q$". I call a such decomposition of an odd integer: "Lemoine-Levy decomposition".

Remark: That is algebraically, $2n + 1 = p + 2q$ has, always, a solution in primes p and q (not necessarily distinct) for $n > 2$.

History: This conjecture was announced by the French mathematician Emile Michel Hyacinthe Lemoine (12/11/1840-21/2/1912) in 1894 [9], and was pondered, in 1963 [10], by the Scottish mathematician Hyman Levy (1889-1975) observing that: 7=3+2× 2, 9=3+2× 3, 11=5+2× 3, 13=7+2× 3, $15 = 5 + 2 \times 5, 17 = 7 + 2 \times 5, 19 = 5 + 2 \times 7, \ldots.$

Remark: This conjecture is a stronger version of the weak Goldbach conjecture saying that: « $\forall n \geq 3 \exists p, q, r$ three prime integers such that: 2n+1=p+q+r "(A prime may be used more than once in the same sum.)

This conjecture is called "weak" because if Goldbach strong conjecture (concerning sums of two primes) is proven, then this would also be true.

"In 2013, Harald Helfgott published (in [6]) a proof of Goldbach's weak conjecture. As of 2018, the proof is widely accepted in the mathematics community, but it has not yet been published in a peer-reviewed journal. The proof was accepted for publication in the Annals of Mathematics Studies series in 2015, and has been undergoing further review and revision since" (See [22]).

In 1999, Dann Corbit has verified the Lemoine- Levy conjecture up to $n \leq 10^9$ (See [1]).
In 1985, John Kiltinen and Peter Young conjectured, in [8], the « refined Lemoine conjecture » extending the Lemoine conjecture. The refined Lemoine conjecture says that: « For any odd number m which is at least 9, there are odd prime numbers p, q, r and s and positive integers j and k such that $m = 2p + q$, $2 + pq = 2^j + r$ and $2q + p = 2^k + s$ ».

Remark: According to [18]: « the study has directed our attention to more subtle aspects of the additive theory of prime numbers. Our conjecture reflects this, dealing with interactions of sums involving primes whereas Goldbach's conjecture and Lemoine's conjecture deal with such sums only individually. This conjecture and the open questions about numbers at levels two and three are of interest in their own right because of the issues they raise within this fascinating and often baffling additive realm of the prime numbers ».

In 2008, the Chinese mathematician Zhi-Wei-Sun announced (see [12]) a similar conjecture saying that:
"$\forall n \geq 1 \exists p$ a prime integer $\exists x \in \mathbb{N}^*$such that 2n+1= p+$x$($x$+1)".

In June 2019, the blog post" makebrainhappy" (see [11]) claimed to have verified the Lemoine-Levy conjecture up to 10^{10}.

For further informations see [2], [5] and [7].

The note: my purpose in the present brief note is to show the Lemoine-Levy conjecture by using, essentially, the elementary topological properties satisfied by the integer part function. The main result of the paper is:

Theorem: $\forall n \in \mathbb{N}, n \geq 3, \exists(p,q)$ two prime integers such that: $2n+1 = p+2q$.
Methods: Considering for $n \geq 3$, the sets:

*$A_n = \{t \in [p_n+1, p_{n+1}], \exists p,q \in \mathbb{P}$: such that $3 \leq p_n + 1 \leq \frac{p+2q-1}{2} \leq p_{n+1}$ and $E(t) = E(t+\frac{1}{2}) = \frac{p+2q-1}{2}\}$
*and $B_n = \{t \in [p_n+1, p_{n+1}],\ [p_n+1, t] \subset A_n\}$,

I show that:$A_n = [p_n+1, p_{n+1}]$, where $(p_n)_{n\geq1}$ is the strictly increasing sequence of prime integers.

So: $[2,+\infty[= \cup_{n=1}^{+\infty}[p_n, p_{n+1}[\ \Rightarrow \forall n \geq 3 \exists(p,q)$ two prime integers such that: $2n+1 = p+2q$.

Organization of The paper: The paper is organized as follows. The §1 is an introduction containing the necessary definition and some History. The §2 contains the ingredients of the proofs. The §3 contains the proof of our main result. The §4 contains the references of the paper given for further reading.

INGREDIENTS OF THE PROOFS

Notation: the closed, the semi-open and the open intervals of $\mathbb{R}$, are (respectively) denoted as below (if $a < b$):
$[a,b]=\{t \in \mathbb{R}, a \leq t \leq b\}$,$[a,b[=\{t \in \mathbb{R}, a \leq t < b\},\}$,$]a,b]=\{t \in \mathbb{R}, a < t \leq b\}$, $]a,b[=\{t \in \mathbb{R}, a < t < b\}$
Remark that: [a, a] = {a}

Definition2: The absolute value function ||is defined on $\mathbb{R}$ by $|t| = \begin{cases} t \ if \ t > 0 \\ 0 \ if \ t = 0 \\ -t \ if \ t < 0 \end{cases}$

Definition3: ([21]) a positive integer p is prime if its set of divisors is $D(p) = \{1,p\}$. The set of all prime integers is denoted $\mathbb{P}$. We define, for $t \geq 2$, the finite set: $\mathbb{P}_t = \{p \in \mathbb{P}, p \leq t\}$, having the cardinal (the number of elements): $card(\mathbb{P}_t) = \pi(t)$ called the prime-counting function.

Proposition1: (Euclid [3]) the set $\mathbb{P}$ of prime integers is a strictly increasing infinite sequence $(p_n)_{n\geq1} = (2,3,5,7,11,13,17,\dots)$

Proposition2: ([21]) we have:$[2,+\infty[= \cup_{n=1}^{+\infty}[p_n, p_{n+1}[$

Définition4: ([13]) we note, for $x \in \mathbb{R}$, by $E(x) \in \mathbb{Z}$ the integer part of the real x i.e. the single integer such that: $E(x) \leq x < E(x)+1$

Proposition3: ([13]) we have:
(i)$\forall x \in \mathbb{Z}: E(x) = x$
(ii)$\forall x \in \mathbb{R}-\mathbb{Z}\ \ E(-x) = -E(x) - 1$
(iii) $0 \leq x < 1 \Rightarrow E(x) = 0$
(iv)$\forall x \in \mathbb{R}: 0 \leq x - E(x) < 1$
(v) $\forall x,y \in \mathbb{R}\ E(x+y) = E(x) + E(y) + \chi_{[1,2[}(x - E(x) + y - E(y))$

Where: $\chi_{[1,2[}(t) = \begin{cases} 1 \ if \ t \in [1,2[\\ 0 \ if \ t \notin [1,2[\end{cases}$ is the characteristic function of the interval [1, 2[
In particular:$\forall x \in \mathbb{Z} \forall y \in \mathbb{R}\ E(x+y) = x + E(y)$

Example: $E\left(x+\frac{1}{2}\right)=E(x)+\chi_{[1,2[}\left(\frac{1}{2}+x-E(x)\right)=E(x) or\ E(x)+1$
(vi)$\forall x,y\in\mathbb{R}\ x<y\Rightarrow E(x)\leq E(y)$

Definition5: If A is any subset of any set X, we define $A^c=\{x\in X, x\notin A\}$ called the complementary set of A.

Definition 6: ([14]) (1) A topological space X is a set equipped with a part :$\tau\subset P(X)$(the set of the parts of X), called topology, such that:
(1)(i)$X,\emptyset\in\tau$
(ii) Any arbitrary (finite or infinite) union of members of τ (i.e. open subsets) still belongs to τ $(i.e\ is\ open)$
(iii) The intersection of any finite number of members of τ $(i.e.$ open subsets)still belongs to τ (I.e. is open)
(2)An element U of τ is called an open subset of X
(3)For $U\in\tau$: U^c is called a closed subset of X

Proposition4: ([14]) in $(\mathbb{R},||)$ (and generally in a metrical space): U open $\Leftrightarrow\forall x\in U\exists\epsilon(x)>0$ such that: $[x-\epsilon(x),x+\epsilon(x)]\subset U$

Proposition 5: ([14]) we have:
(i) Any arbitrary (finite or infinite) intersection of closed subsets of X is still closed.
(ii) The union of any finite number of closed subsets is still closed.

Definition7: ([16]) we call adherence of a subset A of a topological space X, noted $\bar{A}$ the set:

$$\bar{A}=\bigcap_{all\ the\ closed\ subsets\ of\ X\supset A} F$$

If X is a metrical space $\bar{A}=\{a\in X,\exists a_n\in A: a=\lim_{n\to+\infty} a_n\}$
If Y is a subspace of X (equipped with the induced topology) and $A\subset Y$, then:
The adherence of A relatively to Y is $=Y\cap\bar{A}$

Example: if $[a,b]\subset[c,d]$, the adherence of [a, b [relatively to [c, d] is = [a, b]

Proposition 6: ([15]) (i) $A\subset\bar{A}$(ii) $\bar{X}=X,\bar{\emptyset}=\emptyset$(iii)$\overline{\bigcup_{k=1}^m A_k}=\bigcup_{k=1}^m\overline{A_k}$ (iv) $A\subset B\Rightarrow\bar{A}\subset\bar{B}$
(v)A Closed$\Leftrightarrow\bar{A}=A$

Definition8: ([16]) a function $f: X\to Y$ between two metrical spaces X,Y is continuous in t $\in X\Leftrightarrow(\forall(t_n)_n\subset X: (\lim_{n\to+\infty} t_n=t)\Rightarrow(\lim_{n\to+\infty} f(t_n)=f(t)))$

Proposition 7: ([13]) the function E of integer part of a real number is continuous on the set: $\mathbb{R}-\mathbb{Z}$ (the complementary in $\mathbb{R}\ of\ \mathbb{Z}$)

Proposition8: ([17])* any real non empty subset bounded by above A has a supremum: sup $(A)\in\bar{A}$.
*sup (A) is the smallest above bound.

Proposition9: (negation of a proposition [19]) the negation of a proposition (P), denoted non (P), is the proposition true when (P) is false and false when (P) is true. We have: non (non (P)) = (P)

Example: non $(\forall)=\exists$, non $(\exists)=\forall$, non $(=)=\neq$, non $(<)=\geq$

PROOF OT THE LEMOINE-LEVY CONJECTURE

Theorem: $\forall n$ integer $\geq 3\ \exists(p,q)$ prime integers such that: $2n+1=p+2q$

Proof: (of the theorem)
The Proof of the theorem will be deduced from the claims below.

Definition9: For $n\in\mathbb{N}, n\geq 1$, let

*$A_n = \{t \in [p_n+1, p_{n+1}], \exists p,q \in \mathbb{P}$ such that: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1} and\ E(t) = E\left(t+\frac{1}{2}\right) = \frac{p+2q-1}{2}\}$
*$B_n = \{t \in [p_n+1, p_{n+1}],\ [p_n+1, t] \subset A_n\}$
*And $C_n = A_n^c$

Claim1: We have:

$$C_n = \{t \in [p_n+1, p_{n+1}] \text{ such that: } \forall p,q \in \mathbb{P}: 3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1} \begin{cases} \mathrm{E}\left(\mathrm{t}+\frac{1}{2}\right) = \mathrm{E}(\mathrm{t})+1 \\ \text{or } |\mathrm{E}\left(\mathrm{t}+\frac{1}{2}\right) - \frac{\mathrm{p}+2\mathrm{q}-1}{2}| \ge 1\} \\ \text{or } |\mathrm{E}(\mathrm{t}) - \frac{\mathrm{p}+2\mathrm{q}-1}{2}| \ge 1 \end{cases}$$

Proof: (of claim1)
*The result is evident by taking the negation of the relation defining the set: A_n.
*Indeed, we have:
$t \in C_n \Leftrightarrow non(\exists p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}$such that $E(t) = E\left(t+\frac{1}{2}\right) = \frac{p+2q-1}{2})$
$\Leftrightarrow$
$non(\exists p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}$such that $E(t) = E\left(t+\frac{1}{2}\right)$ and $E(t) = \frac{p+2q-1}{2}$ and $E\left(t+\frac{1}{2}\right) = \frac{p+2q-1}{2})$
$\Leftrightarrow non(\ E(t) = E\left(t+\frac{1}{2}\right)$ and $\exists p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}$such that $E(t) = \frac{p+2q-1}{2}$ and $E\left(t+\frac{1}{2}\right) = \frac{p+2q-1}{2})$
$\Leftrightarrow (\ E(t) \ne E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}\ E(t) \ne \frac{p+2q-1}{2}$ or $E\left(t+\frac{1}{2}\right) \ne \frac{p+2q-1}{2})$
$\Leftrightarrow (\ E(t)+1 = E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}\ \left|E(t) - \frac{p+2q-1}{2}\right| > 0$ or $\left|E\left(t+\frac{1}{2}\right) - \frac{p+2q-1}{2}\right| > 0)$
$\Leftrightarrow (\ E(t)+1 = E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1}\ \left|E(t) - \frac{p+2q-1}{2}\right| \ge 1$ or $\left|E\left(t+\frac{1}{2}\right) - \frac{p+2q-1}{2}\right| \ge 1)$

Claim2: We have: $p_n+1 \in A_n$ and $p_n+1 \in B_n$, so: $A_n \ne \emptyset\ and\ B_n \ne \emptyset$

Proof: (of claim2)
For:$n \ge 1$, $2E(p_n+1)+1 = 2E\left(p_n+1+\frac{1}{2}\right)+1 = 2p_n+3$, with $3, p_n \in \mathbb{P} \Rightarrow p_n+1 \in A_n \Rightarrow \{p_n+1\} = [p_n+1, p_n+1] \subset A_n \Rightarrow p_n+1 \in B_n$

Claim3: A_n is closed in $[p_n+1, p_{n+1}]$

Proof: (of claim3)
*Let $(t_m)_m \subset A_n$ converging to $t \in [p_n+1, p_{n+1}]$,and show that $t \in A_n$
*We have:
**$(t_m)_m \subset A_n \Rightarrow E(t_m) = E\left(t_m+\frac{1}{2}\right) = \frac{x_m+2y_m-1}{2}$ with $x_m, y_m \in \mathbb{P}: 3 \le p_n+1 \le \frac{x_m+2y_m-1}{2} \le p_{n+1}$
**$x_m, y_m \in \mathbb{P}$ being bounded above, respectively, by $2p_{n+1}$ and p_{n+1}, we have:

$$\exists N \in \mathbb{N} \exists p,q \in \mathbb{P}\ 3 \le p_n+1 \le \frac{p+2q-1}{2} \le p_{n+1} \text{ Such that:} x_m = p\ and\ y_m = q \forall m \ge N$$

First case: if $t \notin \mathbb{N}$ (so the integer part function$E$ is continuous in t)

$$\begin{cases} E(t_m) = \frac{p+2q-1}{2} \quad \forall m \ge N \\ \lim\limits_{m\to+\infty} t_m = t \notin \mathbb{N} \\ E \text{ continuous in } t \end{cases} \Rightarrow \frac{p+2q-1}{2} = \lim_{m\to+\infty} E(t_m) = E(\lim_{m\to+\infty} t_m) = E(t)$$

Second case: if $t \in \mathbb{N}$ (so E is continuous in: $t+\frac{1}{2} \notin \mathbb{N}$)

$$\begin{cases} E\left(t_m+\frac{1}{2}\right)=\frac{p+2q-1}{2} \quad \forall m \geq N \\ \lim\limits_{m\to+\infty} t_m = t \in \mathbb{N} \\ E \text{ continuous in } t+\frac{1}{2} \end{cases}$$

$\Rightarrow \frac{p+2q-1}{2} = \lim\limits_{m\to+\infty} E\left(t_m+\frac{1}{2}\right) = E(\lim\limits_{m\to+\infty} t_m + \frac{1}{2}) = E(t+\frac{1}{2})$

*So, we have: $\exists p,q \in \mathbb{P}\ 3 \leq p_n+1 \leq \frac{p+2q-1}{2} \leq p_{n+1}$ such that $E(t) = E\left(t+\frac{1}{2}\right) = \frac{p+2q-1}{2}$
*So: the claim3 is showed.

Claim4: B_n has a supremum $\sup(B_n) = \alpha(n)$

Proof: (of claim 4)
*By definition of B_n this set is bounded above (by p_{n+1}) and by claim 2 is non empty
*So: the result follows by combination of proposition8 and claim2.

Claim5: We have: $[p_n+1, \alpha(n)[\subset B_n$

1q2`2222 `1Proof: (of claim5)
*Let $l \in [p_n, \alpha(n)[$
*By definition of $\alpha(n) = \sup(B_n): \exists x \in B_n$ such that: $l \leq x$
*Indeed, if not, we have: $\forall x \in B_n\ l > x$, i.e. l is an above bound of B_n
*So:$\alpha(n)$ being, by proposition 8, the smallest above bound, we have: $l \geq \alpha(n)$
*This contradicting our hypothesis"$l \in [p_n+1, \alpha(n)[$", the result follows.

Claim6: We have: $[p_n+1, \alpha(n)[\subset A_n$

Proof: (of claim 6)
*Let $l \in [p_n+1, \alpha(n)[$
*By claim 5: $l \in B_n$
*So, by definition of B_n, $[p_n+1, l] \subset A_n$
*In particular: $l \in A_n$
*That is: $[p_n+1, \alpha(n)[\subset A_n$
*This ends the proof of claim 5.

Claim7: we have: $[p_n+1, \alpha(n)] \subset A_n$

Proof: (of claim 7)
*By claim 6, we have: $[p_n+1, \alpha(n)[\subset A_n$
*But, by of claim 3, A_n is closed in $[p_n+1, p_{n+1}]$.
*So, by the example following definition 7 and the assertion (v) of proposition6, $\overline{A_n} = A_n \Rightarrow \overline{[p_n+1, \alpha(n)[} = [p_n+1, \alpha(n)] \subset \overline{A_n} = A_n$
*The result follows.

Claim 8: We have: $B_n = [p_n+1, \alpha(n)]$

Proof: (of claim 8)
*By combination of claim5 and claim7, we have: $[p_n+1, \alpha(n)] \subset B_n$
*But by definition of $\alpha(n): l \in B_n \Rightarrow p_n+1 \leq l \leq \alpha(n)$
*That is: $B_n \subset [p_n+1, \alpha(n)]$
*The result follows.

Claim9: If $\alpha(n) < p_{n+1}$, then $\exists h \in]0, p_{n+1} - \alpha(n)]$ such that: $[\alpha(n), \alpha(n)+h] \subset A_n$

Proof: (of claim 9)
*Suppose contrarily that: $\forall h > 0\ [\alpha(n), \alpha(n)+h]$ is not contained in A_n

*That is: $\forall h > 0 \exists x(h) \in [\alpha(n), \alpha(n) + h]$, such that $x(h) \notin A_n$ i.e. $x(h) \in C_n = A_n^c$
*So: $\forall h > 0 \exists y(h) \in]0, h]$ such that: $x(h) = \alpha(n) + y(h) \in C_n$
*By definition of C_n (according to claim 1)) we have:
$E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E\big(\alpha(n) + y(h)\big) + 1 \; or \; \forall p, q \in \mathbb{P} \; 3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1}$:
$|E(\alpha(n) + y(h)) - \frac{p+2q-1}{2}| \ge 1 \; or \; |E(\alpha(n) + y(h) + \frac{1}{2}) - \frac{p+2q-1}{2}| \ge 1$

<u>**First case:**</u> $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E\big(\alpha(n) + y(h)\big) + 1$

<u>**First under-case:**</u> if $\alpha(n) \in \mathbb{N}$
*We have: $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = \alpha(n) + E\left(y(h) + \frac{1}{2}\right) = \alpha(n) + 0 = \alpha(n) = E\big(\alpha(n) + y(h)\big) + 1 = \alpha(n) + E\big(y(h)\big) + 1 = \alpha(n) + 0 + 1 = \alpha(n) + 1$
*This being impossible: the first under case cannot occur.

<u>**Second under-case:**</u> if $\alpha(n) \notin \mathbb{N}$ (so the function E is continuous in $\alpha(n)$)

<u>**The first case under the second under-case:**</u> if $\alpha(n) + \frac{1}{2} \in \mathbb{N}$
$* \; E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = \alpha(n) + \frac{1}{2} + E\big(y(h)\big) = \alpha(n) + \frac{1}{2} + 0 = E(\alpha(n) + \frac{1}{2}) = E\big(\alpha(n) + y(h)\big) + 1 = E\big(\alpha(n)\big) + 1$
*This contradicting claim7 assuring that: $\alpha(n) \in A_n$ (i.e.: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))$) this case cannot occur.

<u>**The second case under the second under-case:**</u> if $\alpha(n) + \frac{1}{2} \notin \mathbb{N}$ (so the function E is continuous in $\alpha(n) + \frac{1}{2}$)
*By tending: $h \to 0$ (noting that, then: $y(h) \to 0$)in the following relation:

$$E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E\big(\alpha(n) + y(h)\big) + 1$$

we have: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))) + 1$
*This contradicting claim7 assuring that: $\alpha(n) \in A_n$ (i.e.: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))$) this case cannot occur.
*So the second under-case cannot, also, occur, because the two possible under-cases are impossible.

<u>**Conclusion:**</u> the first-case cannot occur because the two possible under-cases are impossible.

<u>**Second case:**</u> $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E\big(\alpha(n) + y(h)\big)$
We have: $\forall p, q \in \mathbb{P} \; 3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1} \; |E(\alpha(n) + y(h)) - \frac{p+2q-1}{2}| = |E(\alpha(n) + y(h) + \frac{1}{2}) - \frac{p+2q-1}{2}| \ge 1$

<u>**First under- case:**</u> if $\alpha(n) \notin \mathbb{N}$ (so: E is continuous in $\alpha(n)$)
*By the assertion (i) of proposition 7: $f(t) = E(t)$ is continuous on $\alpha(n)$, so: $\lim_{h \to 0} y(h) = 0 \Rightarrow \forall p, q \in \mathbb{P} \; 3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1} \lim_{h \to 0} |E\big(\alpha(n) + y(h)\big) - \frac{p+2q-1}{2}| = |E\big(\alpha(n)\big) - \frac{p+2q-1}{2}| \ge 1$
*This contradicting claim 7 (assuring that: $\exists p, q \in \mathbb{P}$ such that $3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1}$and $E\big(\alpha(n)\big) = \frac{p+2q-1}{2}$), this case cannot occur.

<u>**Second under- case:**</u> if $\alpha(n) \in \mathbb{N}$, we have:
$\forall p, q \in \mathbb{P} \; 3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1} \; |E\big(\alpha(n) + y(h)\big) - \frac{p+2q-1}{2}| = \left|\alpha(n) + E\big(y(n)\big) - \frac{p+2q-1}{2}\right| = |E\big(\alpha(n)\big) - \frac{p+2q-1}{2}| \ge 1$
*This contradicting claim 7 (assuring that: $\exists p, q \in \mathbb{P}$ such that $3 \le p_n + 1 \le \frac{p+2q-1}{2} \le p_{n+1}$and $E\big(\alpha(n)\big) = \frac{p+2q-1}{2}$), this case cannot occur.
*This being impossible, the second case cannot, also, occur.

Conclusion: The two possible cases coulding not both occur, our starting absurd hypothesis "$\forall h > 0\ [\alpha(n), \alpha(n) + h]$ is not contained in A_n" is not true, so its negation:" $\exists h \in]0, p_{n+1} - \alpha(n)]$ such that: $[\alpha(n), \alpha(n) + h] \subset A_n$" is true.

Claim10: $\alpha(n) = p_{n+1}$

Proof: (of claim10)
*Suppose contrarily that:$\alpha(n) < p_{n+1}$.
*By claim 9: $\exists h \in]0, p_{n+1} - \alpha(n)]$such that: $[\alpha(n), \alpha(n) + h] \subset A_n$
*So, by claim7 and claim 9, we have:
$$[p_n + 1, \alpha(n)] \subset A_n \text{ and } [\alpha(n), \alpha(n) + h] \subset A_n \Rightarrow [p_n + 1, \alpha(n) + h] = [p_n + 1, \alpha(n)] \cup [\alpha(n), \alpha(n) + h] \subset A_n$$
*That is, by definition of B_n, $\alpha(n) + h \in B_n$
*But, by claim 8: $B_n = [p_n + 1, \alpha(n)]$
*So: $\alpha(n) + h \in [p_n + 1, \alpha(n)]$ for $h > 0$ is impossible.

Conclusion: so: our absurd starting hypothesis «$\alpha(n) < p_{n+1}$ » is false and its negation «$\alpha(n) = p_{n+1}$. » is true.

RETURN TO THE PROOF OF THE THEOREM

*By combination of claim8 and claim10, we have:$\forall n$ integer ≥ 1 $[p_n + 1, p_{n+1}] = B_n$

Remark: This means that: $\forall n \geq 2$: $\exists p, q \in \mathbb{P}$ such that: $2p_n + 1 = p + 2q$
*But by the assertion (ii) of proposition2:
$\forall n$ integer ≥ 1 $\exists \varphi(n) \in \mathbb{N}^*, \varphi(n) \geq 2$ such that: $n \in [p_{\varphi(n)}, p_{\varphi(n)+1}[$
*If n is prime: $n = p_{\varphi(n)}$ satisfies the Lemoine-Levy conjecture (By the precedent remark)
$$\forall n \geq 2\ \exists (p, q) \text{ Two prime integers such that:} 2n = p + q$$
*If n is not prime: $\in [p_{\varphi(n)} + 1, p_{\varphi(n)+1}[\subset B_{\varphi(n)}$, and it satisfies, by definition of $B_{\varphi(n)}$, the Lemoine-Levy conjecture.
*So $\forall n \geq 3\ \exists (p, q)$ Two prime integers such that:$2n + 1 = p + 2q$:
*This ends the proof of the Lemoine-Levy conjecture

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