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A- M-QUASI NORMAL OPERATORS IN SEMI HILBERTIAN SPACES

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ABSTRACT

In this paper we introduce the concept of A-*M*-quasi normal operators acting on semi Hilbertian space H with inner product $\langle .,. \rangle$. The object of this paper is to study conditions on T which imply A-*M*-quasi normality. If S and T are A-*M*-quasi normal operators, we shall obtain conditions under which their sum , difference and product are A-*M*-quasi normal.

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Key words: A- adjoint, A -Normal, Semi inner product, *A*-quasinormal and *N*-quasi normal.

INTRODUCTION

Throughout this paper H denotes a complex Hilbert space with inner product $\langle .,. \rangle$ and the norm $\| \,.\, \|$. $L(H)$ stands the Banach algebra of all bounded linear operators on H. $I = I_H$ being the identity operator and if $V \subset H$ is a closed subspace, P_V is the orthogonal projection onto V .

$L(H)^+$ is the cone of positive operators, i.e. $L(H)^+ = \{A \in L(H) : \langle Ax, x \rangle \geq 0, \forall x \in H\}$.

Any positive operator $A \in L(H)^+$ defines a positive semi-definite sesquilinear form

$$\langle .,. \rangle_A : H \times H \to C, \langle x, y \rangle_A = \langle Ax, y \rangle.$$

By $\| \,.\, \|_A$ we denote the semi norm induced by $\langle .,. \rangle_A$ i.e., $\| x \|_A = \langle x | x \rangle^{\frac{1}{2}}$. Note that $\|x\|_A = 0$ if and only if $x \in N(A)$ .Then $\| \,.\, \|_A$ is a norm on H if and only if A is an injective operator, and the semi - normed space $(L(H), \|.\|_A)$ is complete if and only if $R(A)$ is closed. Moreover $\langle .,. \rangle_A$ induces a semi norm on the subspace $\{T \in L(H) | \exists c > 0, \|Tx\|_A \leq c\|x\|_A, \forall x \in H \}$. For this subspace of operators it holds

$$\|T\|_A = \sup_{x \in \overline{R(A)},\, x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty$$

Moreover $\|T\|_A = \sup \left\{ \left| \langle Tx, y \rangle_A \right| ; x, y \in H \text{ and } \|x\|_A \leq 1, \|y\|_A \leq 1 \right\}$.

For $x, y \in H$, we say that x and y are A-orthogonal if $\langle x, y \rangle_A = 0$.

The following theorem due to Douglas will be used (for its proof refer [5].)

Theorem 1.1 Let $T, S \in L(H)$. The following conditions are equivalent.

(i) $R(S) \subset R(T)$.

(ii) There exists a positive number λ such that $SS^* \leq \lambda TT^*$.

(iii) There exists $W \in L(H)$ such that $TW = S$.

From now on, A denotes a positive operator on H ($i.e. A \in L(H)^+$).

Definition 1.2 Let $T \in L(H)$, an operator $W \in L(H)$ is called an A-adjoint of T if

$\langle Tu, v\rangle_A = \langle u, Wv\rangle_A$ for every $u, v \in H$, or equivalently $AW = T^*A$

T is called A- selfadjoint if $AT = T^*A$ and it is called A-positive if AT is positive.

By Douglas Theorem, an operator $T \in L(H)$ admits an A-adjoint if and only if $R(T^*A) \subset R(A)$ and if W is an A-adjoint of T and $AZ = 0$ for some $Z \in L(H)$ then $W + Z$ is also an A-adjoint of T. Hence neither the existence nor the uniqueness of an A -adjoint operator is quaranteed. In fact an operator $T \in L(H)$ may admit none,one or many A-adjoints. From now on, $L_A(H)$ denotes the set of all $T \in L(H)$ which admit an A-adjoint,i.e.

$$L_A(H) = \{T \in L(H) : R(T^*A) \subset R(A)\}$$

$L_A(H)$ is a subalgebra of $L(H)$ which is neither closed nor dense in $L(H)$.

On the other hand the set of all A-bounded operators in $L(H)$ (i.e. with respect the semi norm $\|\ .\ \|_A$ is

$$L_{A^{\frac{1}{2}}}(H) = \left\{T \in L(H) : T^*R(A^{\frac{1}{2}}) \subset R(A^{\frac{1}{2}})\right\} = \left\{T \in L(H) : R(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}) \subset R(A)\right\}$$

Note that $L_A(H) \subset L_{A^{\frac{1}{2}}}(H)$, which shows that if T admits an A-adjoint then it is A-bounded.

If $T \in L(H)$ with $R(T^*A) \subset R(A)$, then T, admits an A-adjoint operator, Moreover there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e. $T^\# = A^+T^*A$, where A^+ is the Moore-Penrose inverse of . The A-adjoint operator $T^\#$ verifies

$$AT^\# = T^*A, R(T^\#) \subseteq \overline{R(A)} \text{ and } N(T^\#) = N(T^*A).$$

In the next we give some important properties of $T^\#$ without proof (refer [4] and [5]).

Theorem 1.3 Let $T \in L_A(H)$. Then

(1) If $AT = TA$ then $T^\# = PT^*$.

(2) $T^\#T$ and $TT^\#$ are A-self adjoint and A-positive.

(3) $\|T\|_A^2 = \|T^\#\|_A^2 = \|T^\#T\| = \|TT^\#\| = w_A(T^\#T) = w_A(TT^\#)$.

(4) $\|S\|_A = \|T^\#\|_A$ for every $S \in L(H)$ which is an A-adjoint of T.

(5) If $S \in L_A(H)$ then $ST \in L_A(H)$, $(ST)^\# = T^\#S^\#$ and $\|TS\|_A = \|ST\|_A$.

(6) $T^\# \in L_A(H), (T^\#)^\# = PTP \text{ and } ((T^\#)^\#)^\# = T^\#$.

(7) $T^{\#^\#} = PTP$

Definition 1.4 An operator $T \in L_A(H)$ is called A -normal if $T^\#T = TT^\#$ (for more details refer [1]).

Definition 1.5 An operator $T \in L_A(H)$ is called A-quasi normal if T commutes with $T^{\#}T$

$$T(T^{\#}T) = (T^{\#}T)T$$. (Refer [7]).

Definition 1.6 An operator $T \in L(H)$, is called *A*-quasi normal if $T(T^{*}T) = N((T^{*}T)T)$ (Refer [8]).

A-*M*-QUASINORMAL OPERATORS

Definition 2.1 An operator $T \in L_A(H)$ is called A-*M*-quasi normal if T commutes with $T^{\#}T$

i.e., $T(T^{\#}T) = M[(T^{\#}T)T]$, where $M > 0$.

Let $T = U + V \in L_A(H)$ where $U = \frac{T + T^{\#}}{2}$ and $V = \frac{T - T^{\#}}{2}$. We shall write $B^2 = TT^{\#}$

and $C^2 = T^{\#}T$ where B and C are non-negative definite .

We give necessary and sufficient conditions for an operator to be A -*M*-quasi normal.

Theorem 2.2 If T is an operator such that (i) B commutes with U and V (ii) $TB^2 = M[C^2T]$ Then T is *A*-*M*-quasi normal operator.

Proof: Since $BU = UB$ and $BV = VB$ we have $B^2U = UB^2$ and $B^2V = VB^2$

Then $B^2T + B^2T^{\#} = TB^2 + T^{\#}B^2$

$$B^2T - B^2T^{\#} = TB^2 - T^{\#}B^2$$

This gives $B^2T = TB^2 = M[C^2T]$.

Hence T is A-*M*-quasi normal operator.

Theorem 2.3 T is A-*M*-quasi normal with $N(A)$ is invariant subspace for T if and only if

C commutes with U and V.

Proof: Since $N(A)$ is invariant subspace for T we observe that $PT = TP$ and $T^{\#}P = PT^{\#}$.

Let T be A-*M*-quasi normal then $T(T^{\#}T) = M[(T^{\#}T)T]$

$$T^{\#}T^{\#^{\#}}T^{\#} = M[T^{\#}(T^{\#}T^{\#^{\#}})]$$

$$T^{\#}PTPT^{\#} = M[T^{\#}T^{\#}PTP]$$

$$PT^{\#}PTT^{\#} = M[T^{\#}PT^{\#}PT]$$

$$T^{\#}TT^{\#} = M[T^{\#}(T^{\#}T)] \text{ hence } T^{\#}TT^{\#} = M(T^{\#^2}T).$$

Since T is *A*-*M*-quasi normal operator we have

$$B^2U = \frac{TT^{\#}T + TT^{\#}T^{\#}}{2}$$
$$= \frac{M[T^{\#}TT] + M[T^{\#}T^{\#}T]}{2}$$
$$= \frac{M[T^{\#}TT + T^{\#}T^{\#}T]}{2}$$
$$= \frac{M[\frac{1}{M}TTT^{\#} + \frac{1}{M}T^{\#}TT^{\#}]}{2}$$
$$= \frac{TTT^{\#} + T^{\#}TT^{\#}}{2} = \frac{T + T^{\#}}{2}TT^{\#} = UB^2$$

Hence the proof.

Theorem 2.4 Let T be an operator such that $C^2U = \frac{1}{M}UC^2$, $C^2V = \frac{1}{M}VC^2$ then

T is *A*-*M*-quasi normal operator.

Proof: Since $C^2U = \frac{1}{M}UC^2$, $C^2V = \frac{1}{M}VC^2$

We have $C^2(U + iV) = \frac{1}{M}(U + iV)C^2$ *, hence* $C^2T = \frac{1}{M}TC^2$

Therefore $(T^{\#}T)T = \frac{1}{M}T(T^{\#}T)$ implies $T(T^{\#}T) = M[(T^{\#}T)T]$.

Hence T is *A*-*M*- quasi normal operator.

Theorem 2.5 Let T be *A*-*M*-quasi normal operator, $N(A)$ is invariant subspace for T and $B^2T = \frac{1}{M}(C^2T)$ then (i) $C^2U = \frac{1}{M}UC^2$ (ii) $C^2V = \frac{1}{M}VC^2$.

Proof: Since $B^2T = \frac{1}{M}(C^2T) = (TT^{\#})T = \frac{1}{M}((T^{\#}T)T) \Rightarrow T^{\#}(TT^{\#}) = \frac{1}{M}(T^{\#}(T^{\#}T))$

Since is *A*-*M*-quasi normal operator we have

$$C^2U=\left(T^{\#}T\right)\left(\frac{T+T^{\#}}{2}\right)$$
$$=\frac{T^{\#}TT+T^{\#}TT^{\#}}{2}$$
$$=\frac{1}{M}\left(\frac{TT^{\#}T+T^{\#}T^{\#}T}{2}\right)$$
$$=\frac{1}{M}\left(\frac{T+T^{\#}}{2}\right)\left(T^{\#}T\right)=\frac{1}{M}UC^2$$

(ii) Similarly $C^2V=\frac{1}{M}VC^2$

Theorem 2.6 Let S and T be two A-*M*-quasi normal operators such that $ST=TS=S^{\#}T=T^{\#}S=ST^{\#}=TS^{\#}=0$. Then their sum $S+T$ and difference $S-T$ are A-*M*-quasi normal operators.

Proof: $(S+T)(S+T)^{\#}(S+T)$

$$=(S+T)\left(S^{\#}+T^{\#}\right)(S+T)$$
$$=(S+T)\left(S^{\#}S+S^{\#}T+T^{\#}S+T^{\#}T\right)$$
$$=(S+T)\left(S^{\#}S+T^{\#}T\right)$$
$$=SS^{\#}S+ST^{\#}T+TS^{\#}+TT^{\#}T$$
$$=SS^{\#}S+TT^{\#}T$$
$$=M\left((S^{\#}S)S\right)+M\left((T^{\#}T)T\right)$$
$$=M\{\left((S^{\#}S)S\right)+\left((T^{\#}T)T\right)\}$$
$$=(S+T)^{\#}(S+T)^2$$

Hence $S+T$ is A-quasi normal. Similarly $S-T$ is A-quasi normal.

Theorem 2.7 Let S be A-*M*-quasi normal operator and T be quasi normal operator. Then their product ST is A-*M*-quasi normal if the following conditions are satisfied (i) $ST=TS$

(ii) $ST^{\#}=T^{\#}S$ (iii) $TS^{\#}=S^{\#}T$.

Proof. $(ST)(ST)^{\#}(ST)=(ST)\left(T^{\#}S^{\#}\right)(ST)$

$$=(ST)\left(S^{\#}T^{\#}\right)(ST)$$

$$= S(TS^{\#})(T^{\#}S)T$$

$$= SS^{\#}(TS)T^{\#}T$$

$$= SS^{\#}(ST)T^{\#}T$$

$$= (SS^{\#}S)(TT^{\#}T)$$

$$= M((S^{\#}S)S)(T^{\#}TT)$$

$$= M(SS^{\#}ST^{\#}TT)$$

$$= M(SS^{\#}T^{\#}STT)$$

$$= M(T^{\#}S^{\#}STST)$$

$$= M[(ST)^{\#}(ST)(ST)]$$

Hence ST is A-*M*-quasi normal.

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