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THE AFFIRMATIVE RESOLUTION OF THE LANDAU FOURTH PROBLEM

Mohammed Ghanim
Ecole Nationale de Commerce et de Gestion B.P 1255 Tanger Maroc

ABSTRACT

The present short paper is an affirmative resolution of « the Landau fourth problem », known as « the near-square primes conjecture », remained open since 1912 and saying that : « there are infinitely many primes of the form n^2+1 », by using the topological properties of the integer part function as in my previous paper [8] published by the GJAETS on June 30, 2021 .

KEYWORDS: Prime integer, integer part function, Landau fourth problem, near-square primes conjecture, finite set, infinite set, prime-counting function 2010
Mathematics subject classification: 11 A xx (Elementary Number Theory)

INTRODUCTION

Definition 1: We call « **the Landau fourth problem** » or «the **near-square primes conjecture** » the following assertion « it exists an infinite number of prime integers p such that $\sqrt{p-1} \in \mathbb{N}$ »

Example: The list of known primes of this form is obtained: for n=1, 2, 4, 6, 10, 14, 16, 20…Indeed we have that the numbers: $n^2+1 = 2, 5, 7, 37, 101, 197, 257, 401, \ldots$ are primes.
One example of near-square primes is Fermat primes. Recall, according to [33] that: "In mathematics, a **Fermat number**, named after Pierre de Fermat, who first studied them, is a positive integer of the form:

$$F_n = 2^{2^n} + 1$$

Where *n* is a non-negative integer. The first few Fermat numbers are:

3, 5, 17, 257, 65537, 4294967297, 18446744073709551617 …

If 2^k+1 is prime and $k > 0$, then k must be a power of 2, so 2^k+1 is a Fermat number; such primes are called **Fermat primes**. As of 2021, the only known Fermat primes are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$; heuristics suggest that there are no more".

Remark: It is noted in [28] that « the existence of infinitely many such primes would follow as a consequence of other number-theoretic conjectures such as the Bunyakovsky conjecture and the Bateman-Horn conjecture ».
Recall that the:

* **Bunyakovsky (or Bouniakowsky) conjecture:** according to [30] : " gives a criterion for a polynomial f(x) in one variable with integer coefficients to give infinitely many prime values in the f(1), f(2), f(3),…f(n) sequence. It was stated in 1857 by the Russian mathematician Viktor Bunyakovsky. The following three conditions are necessary for f(n) to have the desired prime-producing property:

1. The leading coefficient is positive.
2. The polynomial is irreducible over the integers.
3. The values f(1), f(2), f(3),… have no common factor. (In particular, the coefficients of f(x) should be relatively prime).

Bunyakovsky conjecture is that these conditions are sufficient: if f(x) satisfies (1)-(3), then f(n) is prime for infinitely many positive integers. A statement that is equivalent to Bunyakovsky conjecture is that for every integer polynomial f(x) that satisfies (1)-(3), f(n) is prime for at least one positive integer n. This can be seen by considering the sequence of polynomials f(t), f(t+1), f(t+2),… etc.. Bunyakovsky conjecture is a special case of Schinzel's hypothesis H, one of the most famous open problems in number theory".

***Dickson's conjecture**: stated by Dickson (1904) says according to [34] that " for a finite set of linear forms $a_1 + b_1n$, $a_2 + b_2n$, ..., $a_k + b_kn$ with $b_i \geq 1$, there are infinitely many positive integers n for which they are all prime, unless there is a congruence condition preventing this. The case $k = 1$ is Dirichlet's theorem [34].

Two other special cases are well-known conjectures: there are infinitely many twin primes (n and $2 + n$ are primes), and there are infinitely many Sophie Germain primes (n and $1 + 2n$ are primes).Dickson's conjecture is further extended by Schinzel's hypothesis H".

***Dickson's generalized conjecture**: says, according to [34], that « Given n polynomials with positive degrees and integer coefficients (n can be any natural number) that each satisfy all three conditions in the Bunyakovsky conjecture, and for any prime p there is an integer x such that the values of all n polynomials at x are not divisible by p, then there are infinitely many positive integers x such that all values of these n polynomials at x are prime. For example, if the conjecture is true then there are infinitely many positive integers x such that $x^2 + 1$, $3x - 1$, and $x^2 + x + 41$ are all prime. When all the polynomials have degree 1, this is the original Dickson's conjecture. This more general conjecture is the same as the Generalized Bunyakovsky conjecture ».

***A prime constellations: A** prime constellation, also called a prime k-tuple, prime k-tuplet, or prime cluster, is, according to [23]: « a sequence of k consecutive numbers such that the difference between the first and last is, in some sense, the least possible. More precisely, a prime k-tuplet is a sequence of consecutive primes (p_1, p_2, ..., p_k) with $p_k - p_1 = s(k)$, where $s(k)$ is the smallest number s for which there exist k integers $b_1 < b_2 < \ldots < b_k$, $b_k - b_1 = s$ and, for every prime q, not all the residues modulo q are represented by b_1, b_2, ..., b_k. For each k, this definition excludes a finite number of clusters at the beginning of the prime number sequence. For example, (97, 101, 103, 107, 109) satisfies the conditions of the definition of a prime 5-tuplet, but (3, 5, 7, 11, 13) does not because all three residues modulo 3 are represented ».

***The first Hardy-Littlewood conjecture:** according to [24]:" The first of the Hardy-Littlewood conjectures, known as the k-tuple conjecture, states that the asymptotic number of prime constellations can be computed explicitly. In particular, unless there is a trivial divisibility condition that stops p, $p + a_1$, ..., $p + a_k$ from consisting of primes infinitely often, then such prime constellations will occur with an asymptotic density which is computable in terms of a_1, ..., a_k. Let $0 < m_1 < m_2 < \ldots < m_k$, then the k-tuple conjecture predicts that the number of primes $p \leq x$ such that $p + 2m_1$, $p + 2m_2$, ..., $p + 2m_k$ are all prime is :

$$\pi_{m_1, m_2, \ldots, m_k}(x) \sim C(m_1, m_2, \ldots, m_k) \int_2^x \frac{dt}{\ln^{k+1} t}, \quad (1)$$

Where :

$$C(m_1, m_2, \ldots, m_k) = 2^k \prod_q \frac{1 - \frac{w(q; m_1, m_2, \ldots, m_k)}{q}}{\left(1 - \frac{1}{q}\right)^{k+1}}, \quad (2)$$

the product is over odd primes q, and :

$$w(q; m_1, m_2, \ldots, m_k) \quad (3)$$

denotes the number of distinct residues of 0, m_1, ..., m_k (mod q) . If $k = 1$, then this becomes:

$$C(m) = 2 \prod_{\substack{q \\ q \text{ prime}}} \frac{q(q-2)}{(q-1)^2} \prod_{q|m} \frac{q-1}{q-2}. \quad (4)$$

This conjecture is generally believed to be true, but has not been proven.
The twin prime conjecture :

$$\pi_2(x) \sim 2\,\Pi_2 \int_2^x \frac{dx}{(\ln x)^2} \quad (5)$$

is a special case of the k-tuple conjecture with $S = \{0, 2\}$, where Π_2 is known as the twin primes constant.

The following special case of the conjecture is sometimes known as the prime patterns conjecture. Let S be a finite set of integers. Then it is conjectured that there exist infinitely many k for which $\{k+s : s \in S\}$ are all prime iff S does not include all the residues of any prime. This conjecture also implies that there are arbitrarily long arithmetic progressions of primes ».

The second Hardy–Littlewood conjecture: saying, according to [36], that : « $\pi(x + y) \leq \pi(x) + \pi(y)$ for $x, y \geq 2$, where $\pi(x)$ denotes the prime-counting function, giving the number of prime numbers up to and including x».

***Schinzel's hypothesis H**, according to [31]: " aims to define the possible scope of a conjecture of the nature that several sequences of the type f(n), g(n),…with values at integers n of irreducible integer-valued polynomials: f(x), g(x),…Should be able to take on prime number values simultaneously, for arbitrarily large integers n. Putting it another way, there should be infinitely many such n for which each of the sequence values are prime numbers. Some constraints are needed on the polynomials. Schinzel's hypothesis builds on the earlier Bunyakovsky conjecture, for a single polynomial, and on the Hardy–Littlewood conjectures and Dickson's conjecture for multiple linear polynomials. It is in turn extended by the Bateman–Horn conjecture. Note that the coefficients of the polynomials need not to be integers; for example, this conjecture includes the polynomial$\frac{1}{2}t^2 + \frac{1}{2}t + 1$, since it is an integer-valued polynomial".

* **Bateman-Horn conjecture**, according to [32]: « provides a conjectured density for the positive integers at which a given set of polynomials all have prime values. For a set of m distinct irreducible polynomials $f_1,..., f_m$ with integer coefficients, an obvious necessary condition for the polynomials to simultaneously generate prime values infinitely often is that they satisfy Bunyakovsky property, that there does not exist a prime number p that divides their product $f(n)$ for every positive integer n. For, if there were such a prime p, having all values of the polynomials simultaneously prime for a given n would imply that at least one of them must be equal to p, which can only happen for finitely many values of n or there would be a polynomial with infinitely many roots, whereas the conjecture is how to give conditions where the values are simultaneously prime for infinitely many n. An integer n is prime-generating for the given system of polynomials if every polynomial $f_i(n)$ produces a prime number when given n as its argument. If $P(x)$ is the number of prime-generating integers among the positive integers less than x, then the Bateman–Horn conjecture states that:

$$P(x) \text{ is equivalent to } \frac{C}{D}\int_2^x \frac{dt}{(\ln(t))^m}$$

Where D is the product of the degrees of the polynomials and where C is the product over primes p:

$$C=\prod_p \frac{1-\frac{N(p)}{p}}{(1-\frac{1}{p})^m}$$

With $N(p)$ the number of solutions to: $f(n) \equiv 0 \; [mod\; p]$

Bunyakovsky propertyimplies $N(p) < p$, for all primes p, so each factor in the infinite product C is positive. Intuitively one then naturally expects that the constant C is itself positive, and with some work this can be proved. (Work is needed since some infinite products of positive numbers equal zero) ».

History: In 1912, the German mathematician Edmund Georg Hermann Landau (1877-1938) listed, in his speech (see [14]) delivered before the Cambridge fifth International congress of mathematicians, four basic problems about prime integers. In this speech Landau said that these problems, now known as « **Landau's problems »,** are « unattackable at the present state of mathematics». These problems are cited below:

1. **The problem1:** about the « Goldbach conjecture » saying: « can every even integer greater than 2 be written as the sum of two primes? »
2. **The problem2:** about « the twin primes conjecture » saying: « are there infinitely many primes p such that $p + 2$ are prime? »
3. **Problem3:** about « the Legendre conjecture » saying: « does there always exist at least one prime between consecutive perfect squares? »

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4. **Problem4**: about « the near-square primes conjecture » saying: « Are there infinitely many primes p such that $p-1$ is a perfect square? » i.e.: « are there infinitely many primes of the form n^2+1? »

Mohammed Ghanim has resolved :

1. **The Goldbach conjecture** : in a paper [7] entitled « confirmation of the Goldbach binary conjecture »published by the GJAETS in May 20-2018, based on the Schoenfeld inequality proved by M.Ghanim in [6] and by a direct proof in a paper [8] entitled "confirmation of the binary Goldbach conjecture by an elementary short proof" published by the GJAETS in June 30- 2021.
2. **The twin primes conjecture**: in a paper [9] entitled « confirmation of the Polignac and the twin primes conjecture » published by the GJAETS in July 10, 2018. The paper is also based on the Schoenfeld inequality [6].
3. **The Legendre conjecture**: in a paper [10] entitled « confirmation of the Legendre and the Euler conjectures » published by the GJAETS in August 10-2018. The paper is also based on the Schoenfeld inequality [6].
4. **The near-square primes conjecture :** by the present paper

So, by resolving the four landau problems, I have finished my Project of resolving the Landau program about prime integers.

"Henryk Iwaniec showed in [13] that there are infinitely many numbers of the form n^2+1 with at most two prime factors "(see [28]).

"Nesmith Ankeny proved in [1] that, assuming the extended Riemann hypothesis for L-functions on Hecke characters, there are infinitely many primes of the form x^2+y^2 with : $y = O(\ln(x))$. Landau's conjecture is for the stronger case: $y = 1$" (see [28]).

"Merikoski, ([17]) improving on previous works, ([3], [4], [11], [12], [22]), showed that there are infinitely many numbers of the form n^2+1 with greatest prime factor at least: $n^{1.279}$. Replacing the exponent with 2 would yield Landau's conjecture" (see [28]).

"The Brun Sieve establishes an upper bound on the density of primes having the form:$p = n^2+1$. There are $O(\frac{\sqrt{x}}{\ln(x)})$ such primes up to x . It then follows that almost all numbers of the form n^2+1 are composite" (see [28]).

Recall that:

The Brun Sieve: is a Number theory technique, developed by Viggo Brun in 1915, for estimating the size of "sifted sets" of positive integers which satisfy a set of conditions expressed by congruences. According to [37]):

"In terms of sieve theory the Brun sieve is of combinatorial *type*; that is, it derives from a careful use of the inclusion–exclusion principle (i.e. for finite sets $A_1, A_2, \dots, A_n$ one has the identity:$|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq J \subset \{1,2,\dots,n\}} (-1)^{|J|+1} |\bigcap_{j \in J} A_j|$).

Let A be a set of positive integers $\leq x$ and let P be a set of primes. For each p in P, let A_p denote the set of elements of A divisible by p and extend this to let A_d the intersection of the A_p for p dividing d, when d is a product of distinct primes from P. Further let A_1 denote A itself. Let z be a positive real number and $P(z)$ denote the primes in $P \leq z$. The object of the sieve is to estimate:$S(A, P(z)) = |A - \bigcup_{p \in P(z)} A_p|$

We assume that $|A_d|$ may be estimated by: $|A_d| = \frac{w(d)}{d} X + R_d$ if X=|A| (the cardinal of A=number of elements of the finite set A)

where w is a multiplicative function (In number theory, a **multiplicative function** is an arithmetic function $f(n)$ of a positive integer n with the property that $f(1) = 1$ and whenever a and b are coprime, then : f(ab)=f(a)f(b). An arithmetic function $f(n)$ is said to be **completely multiplicative** (or **totally multiplicative**) if $f(1) = 1$ and $f(ab) = f(a)f(b)$ holds *for all* positive integers a and b, even when they are not coprime). Let: $W(p)=\prod_{p\in P(z)}(1-\frac{w(p)}{p})$

<u>Brun's pure sieve</u> (See Cojocaru & Murty, Theorem 6.1.2 [2]): With the notation as above, assume that

*$|R_d| \le w(d)$ for any squarefree d composed of primes in P ;

*$w(p) < C$ for all p in P;

- $\sum_{p\in P(z)} \frac{w(p)}{p} < D\log(\log z) + E$

Where C, D, E are constants.

Then: $S(A, P, z) = XW(z)\,(1+O((\log(z))^{-b\log(b)}) + O(z^{b\log(\log z)})$, where b is any positive integer.

In particular, if $\log z < c \log x / \log\log x$ for a suitably small c, then: $S(A, P, z) = XW(z)(1+o(1))$

Applications

- Brun's theorem: the sum of the reciprocals of the twin primes converges;
- Schnirelmann's theorem: every even number is a sum of at most C primes (where C can be taken to be 6);
- There are infinitely many pairs of integers differing by 2, where each of the member of the pair is the product of at most 9 primes;
- Every even number is the sum of two numbers each of which is the product of at most 9 primes.

The last two results were superseded by Chen's theorem, and the second by Goldbach's weak conjecture (C = 3) ‘’ (See [37])

For further History see the references: [16], [18], [19], [20], [21] and [28].

<u>The note:</u> The present short paper is an affirmative resolution of « the Landau fourth problem », known as « the near-square primes conjecture », saying that: « there are infinitely many primes of the form n^2+1 », by using only elementary tools of mathematics such as the continuity of the integer part, the properties of maximums and closed sets.

<u>Results:</u> Our main result is:

<u>Theorem:</u> if $(p_n)_{n\ge1}$ denotes the sequence of prime integers, then the set $A = \{n \in \mathbb{N}^*, \exists m \in \mathbb{N}^* \text{ such that } \mathrm{p_n} = \mathrm{m}^2 + 1\}$ is infinite.

<u>Methods:</u> We proceed by the absurd reasoning.

<u>Organization of The paper:</u> The paper is organized as follows. The §1 is an introduction containing the necessary definition and some History. The §2 contains the ingredients of the proofs. The §3 contains the proof of our main result. The §4 contains the references of the paper given for further reading.

INGREDIENTS OF THE PROOFS

<u>Definition2:</u> (prime integer and prime-counting function [26], [29]) a positive integer p is prime if its set of divisors is: $D(p) = \{1, p\}$. The set of all prime integers is denoted $\mathbb{P}$. For $t \ge 2$, we define the set: $\mathbb{P}_t = \{p \in \mathbb{P}, p \le t\}$ which is a finite set having the cardinal (the number of elements (see [24])): $card(\mathbb{P}_t) = \pi(t)$ called the prime-counting function.

<u>Proposition1:</u> (Euclid theorem [5]) The set $\mathbb{P}$ of prime integers is a strictly increasing infinite sequence$(p_n)_{n\ge1} = (2,3,5,7,11,13,17, \dots)$

<u>Proposition2:</u> ([29]) we have: (1) $t < s \Rightarrow \pi(t) \le \pi(s)$ (2) $\pi(p_n) = n$

<u>Définition3:</u> ([27]) we note, for $x \in \mathbb{R}$, by $E(x) \in \mathbb{Z}$ the integer part of the real x i.e. the single integer such that: $E(x) \le x < E(x) + 1$

Proposition4: ([27]) we have:
(i)$\forall x \in \mathbb{Z}: E(x) = x$
(ii)$\forall x, y \in \mathbb{R}\ x < y \Rightarrow E(x) \leq E(y)$
(iii)$\forall x \in \mathbb{R} - \mathbb{Z}\ \ E(-x) = -E(x) - 1$
(iv) $0 \leq x < 1 \Rightarrow E(x) = 0$
(v)$\forall x \in \mathbb{R}: 0 \leq x - E(x) < 1$
(vi) $\forall x, y \in \mathbb{R}\ E(x + y) = E(x) + E(y) + \chi_{[1,2[}(x - E(x) + y - E(y))$
Where: $\chi_{[1,2[}(t) = \begin{cases} 1\ if\ t \in [1,2[\\ 0\ if\ t \notin [1,2[\end{cases}$ is the characteristic function of the interval [1,2[
In particular:$\forall x \in \mathbb{Z} \forall y \in \mathbb{R}\ E(x + y) = x + E(y)$

Example: $E\left(x + \frac{1}{2}\right) = E(x) + \chi_{[1,2[}\left(\frac{1}{2} + x - E(x)\right) = E(x) or\ E(x) + 1$

Definition 4: (Continuity [39])$f: I(\text{real interval}) \to \mathbb{R}$ continuous in $t \Leftrightarrow (\forall (t_n)_n: \lim_{n \to +\infty} t_n = t \Rightarrow \lim_{n \to +\infty} f(t_n) = f(t))$

Definition5: (The complementary set [38]] If A is any subset of any set X, we define $A^c = X - A = \{x \in X, x \notin A\}$ called the complementary set of A.

Proposition5: ([27]) (1) the integer part function E is continuous on $\mathbb{R} - \mathbb{Z}$
(2) The function $t \to p_{E(t)}$ ($p_{E(t)}$ being the prime integer of order $E(t)$ the integer part of t) is continuous on $\mathbb{R} - \mathbb{Z}$

Definition 6: (Closed set [40])A is closed in $[1, +\infty[\Leftrightarrow (\forall (t_n)_n \subset A\ \lim_{t \to +\infty} t_n = t$ in $[1, +\infty[\Rightarrow t \in A)$

Proposition6: (Maximal element [41]) any non empty closed part A of $[1, +\infty[$ bounded above has a maximal element max (A)$\in A$.

THE AFFIRMATIVE RESOLUTION OF THE LANDAU FOURTH PROBLEM

Theorem: it exists an infinite number of prime integers p such that $\sqrt{p - 1} \in \mathbb{N}$

Proof: (of the theorem)
The proof of the theorem will follow from the lemmas below.

Definition7: Let $A = \{n \in \mathbb{N}^*, \exists m \in \mathbb{N}^*$ such that $\mathrm{p_n} = \mathrm{m}^2 + 1\}$
$B = \{t \in [1, +\infty[$, such that $\exists m \in \mathbb{N}^*\ p_{E(t)} = p_{E(t+\frac{1}{2})} = m^2 + 1\}$
$C = B^c$

Lemma1: We have:
C=$\{t \in [1, +\infty[, E\left(t + \frac{1}{2}\right) = E(t) + 1\ or\ \forall m \in \mathbb{N}^* |p_{E\left(t+\frac{1}{2}\right)} - m^2 - 1| \geq 1 or\ |p_{E(t)} - m^2 - 1| \geq 1\}$

Proof: (of lemma1)
The result follows by taking the negation of the proposition defining the set B.

Lemma2: and $A \neq \emptyset, B \neq \emptyset$ and $C \neq \emptyset$

Proof: (of lemma2)
*The result follows by noting that:$2 = 1^2 + 1$, with 2 prime$\Rightarrow 2 \in A$
*We have: $\forall m \in \mathbb{N}^*\ \left|p_{E(2)} - m^2 - 1\right| = |p_2 - m^2 - 1| = |3 - m^2 - 1| = |m^2 - 2| > 0 \Rightarrow t = 2 \in C$
*$A \subset B \Rightarrow B \neq \emptyset$

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Lemma3: the set B is closed in $[1, +\infty[$

Proof: (of lemma3)
*Let $(t_q)_q \subset B$ such that $\lim_{q\to+\infty} t_q = t \in [1,+\infty[$. Show that: $t \in B$
*By definition of B: $\forall q \in \mathbb{N}^* \ \exists m_q \in \mathbb{N}^*$ *such that* $p_{E(t_q+\frac{1}{2})} = p_{E(t_q)} = 1 + {m_q}^2$

First case: if $t \in \mathbb{N}^*$ (so: E is continuous in $t + \frac{1}{2} \notin \mathbb{N}$)
*So: $\lim_{q\to+\infty} p_{E(t_q+\frac{1}{2})} = p_{E\left(t+\frac{1}{2}\right)} = 1 + \lim_{q\to+\infty} {m_q}^2 = 1 + m^2$

***Remark:**: $(m_q)_q$ being a bounded sequence of integers$\Rightarrow m_q^2 \to m^2 (m \ being \ an \ integer)$ when $q \to +\infty$

Second case: if $t \notin \mathbb{N}^*$(so: E is continuous in $t \notin \mathbb{N}$)
* $\lim_{q\to+\infty} p_{E(t_q)} = p_{E(t)} = 1 + \lim_{q\to+\infty} {m_q}^2 = 1 + m^2$

Conclusion: So, we have : $p_{E(t)} = p_{E\left(t+\frac{1}{2}\right)} = 1 + m^2$, that is $t \in B$

Lemma4: B is not bounded above

Proof: (of lemma5)
*Suppose contrarily that B is bounded above.

Claim1: the set B has a maximal element $\alpha \in B$

Proof: (claim1)
The result follows by proposition6, B being not empty, closed and bounded above

Claim2: $\forall h > 0 \ \ , \alpha + h \in C = B^c$

Proof (of the claim2)
The result follows by definition of the $\alpha = \max(B)$ as a maximum

Claim3: We have:
$\forall h > 0 \quad E\left(\alpha + h + \frac{1}{2}\right) = E(\alpha + h) + 1 \ or \ \forall m \in \mathbb{N}^* |p_{E\left(\alpha+h+\frac{1}{2}\right)} - m^2 - 1| \geq 1 or \ |p_{E(\alpha+h)} - m^2 - 1| \geq 1$

Proof: (of claim3)
The result follows by lemma1 and the definition of the set C

Claim4: the relations of Claim 3 are impossible
Proof: (of claim 4)
Indeed: tending h to zero, we have:

First case: $E\left(\alpha + h + \frac{1}{2}\right) = E(\alpha + h) + 1$

First under-case: if $\alpha \in \mathbb{N}$
*We have:

$$E\left(\alpha + h + \frac{1}{2}\right) = \alpha + E\left(h + \frac{1}{2}\right) = \alpha + 0 = \alpha = E(\alpha + h) + 1 = \alpha + E(h) + 1 = \alpha + 0 + 1 = \alpha + 1$$

*This being impossible: the first under case cannot occur.

Second under-case: if $\alpha \notin \mathbb{N}$ (so the function E is continuous in α)

The first case under the second under-case: if $\alpha + \frac{1}{2} \in \mathbb{N}$

$* E\left(\alpha + h + \frac{1}{2}\right) = \alpha + \frac{1}{2} + E(h) = \alpha + \frac{1}{2} + 0 = E(\alpha + \frac{1}{2}) = \lim_{h \to 0} E(\alpha + h) + 1 = E(\alpha) + 1$

*This contradicting claim7 assuring that: $\alpha \in B$ (i.e.: $E\left(\alpha + \frac{1}{2}\right) = E(\alpha)$) this case cannot occur.

The second case under the second under-case: if $\alpha + \frac{1}{2} \notin \mathbb{N}$ (so the function E is continuous in $\alpha + \frac{1}{2}$)

*By tending:$h \to 0$ in the following relation: $E\left(\alpha + h + \frac{1}{2}\right) = E(\alpha + h) + 1$, we have:

$$E\left(\alpha + \frac{1}{2}\right) = E(\alpha) + 1$$

*This contradicting lemma4 assuring that: $\alpha \in B$ (i.e.: $E\left(\alpha + \frac{1}{2}\right) = E(\alpha)$) this case cannot occur.

*So the second under-case cannot, also, occur, because the two possible under-cases are impossible.

Conclusion: the first-case cannot occur because the two possible under-cases are impossible

Second case: $E\left(\alpha + h + \frac{1}{2}\right) = E(\alpha + h)$

We have$\forall m \in \mathbb{N}^*$ *such that* $|p_{E(\alpha+h)} - m^2 - 1| = |p_{E\left(\alpha+h+\frac{1}{2}\right)} - m^2 - 1| \geq 1$

First under- case: if $\alpha \notin \mathbb{N}$ (so: E is continuous in α)

*By proposition 5: $f(t) = p_{E(t)}$ is continuous on α, so:

$$\lim_{h \to 0} |p_{E(\alpha+h)} - m^2 - 1| = |p_{E(\alpha)} - m^2 - 1| \geq 1$$

*This contradicting claim 1 (assuring that: $\exists m \in \mathbb{N}^*$ $p_{E(\alpha)} = 1 + m^2$), this case cannot occur.

Second under- case: if $\alpha \in \mathbb{N}$, we have:

$\forall m \in \mathbb{N}^*$ $\left|p_{E(\alpha+h)} - m^2 - 1\right| = \left|p_{\alpha+E(h)} - m^2 - 1\right| = |p_\alpha - m^2 - 1| = \left|p_{E(\alpha)} - m^2 - 1\right| \geq 1$

*This contradicting claim 1 (assuring that: $\exists$m $\in \mathbb{N}^*$ $E_{E(\alpha)} = 1 + m^2$), this case cannot occur.

*This being impossible, the second case cannot, also, occur.

RETURN TO THE PROOF OF LEMMA4

Conclusion: The two possible cases coulding not both occur, our starting absurd hypothesis "B is bounded above" is false and so its negation:" B is not bounded above" is true.

Lemma5: We have: $\forall t \in B \exists n(t) \in A$ such that: $t < n(t) + 1$

Proof: (of lemma5)

*It is sufficient to take for $t \in B$: $n(t) = E(t) \in A$

*Indeed:

**$t \in B \Rightarrow \exists m \in \mathbb{N}^*$ such that $p_{E(t)} = m^2 + 1 \Rightarrow E(t) \in A$

**By definition of $E(t)$, we have: $t < E(t) + 1$

*So lemma 5 is proved.

RETURN TO THE PROOF OF THE THEOREM

*Suppose contrarily that the set A is finite

*So: $\exists L > 0$ such that: $\forall n \in A$ $n \leq L$

*But, by lemma5: $\forall t \in B$: $\exists n \in A$ such that: $t < n + 1$

*That is: $\exists L > 0$ such that: $\forall t \in B$ $t < n + 1 \leq L + 1$ i.e. B is bounded above by $L + 1$.

*This contradicting lemma4, the result follows.

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