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# CONFIRMATION OF THE BINARY GOLDBACH CONJECTURE BY AN ELEMENTARY SHORT PROOF

**Mohammed Ghanim**
* Ecole Nationale de Commerce et de Gestion
B.P 1255 Tanger Maroc

Citation
«La loi de Gabor : Tout ce qui est techniquement faisable, possible, sera fait un jour, tôt ou tard »
The Hungarian physicist Dénes Gabor (1900-1979)
www.bruno-jarrosson.com/la-loi-de-gabor/

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## ABSTRACT

I have proved in [3], on May 2018, the Goldbach (1690-1764) binary conjecture, remained open since 1742, saying that any even integer greater than 4 is the sum of two prime integers , by using the Schoenfeld (1920-2002) inequality [9] showed by myself, on April 2017, in [4]. Now I confirm the Goldbach conjecture by using, essentially, some topological properties of the Integer part function.

**KEYWORDS**: prime integer, prime-counting function, Goldbach conjecture, supremum, continuity, upper semi-continuity 2010 Mathematics Subject Classification: A 11 xx (Number theory).

## INTRODUCTION

**Definition 1:** We call « the Goldbach conjecture » or « the Goldbach's strong conjecture » or « the Goldbach binary conjecture » or « the Goldbach problem » (according to D.Hilbert) or « the Goldbach theorem » (according to G.H.Hardy) the following assertion: "any even integer greater than 4 is the sum of two prime integers" that is: " $\forall n$ an integer $\geq 2 \exists (p, q)$ two prime positive integers such that:$2n = p + q$". I call this decomposition of 2n with the summum of two prime positive integers: "Goldbach decomposition".

**History:** the Goldbach conjecture was announced by the German Mathematician Christian Goldbach (1690-1764) in a letter addressed to the Swiss Mathematician Leonhard Euler (1707-1783) in 7 June 1742 [5]. Really Goldbach conjectured, in the letter to Euler, that: "any integer greater than 2 can be written as a sum of three prime integers". In 30 June 1742, Euler reformulated, responding to Goldbach, the conjecture as "any even integer than 4 is the sum of two prime integers" and wrote to Goldbach: "I consider that this is, absolutely, a certain theorem, although I cannot prove it". So the conjecture has remained, since 1742, without a rigorous poof although many attempts by the great mathematicians.

In 1900, the German Mathematician David Hilbert (1862-1943) said in his conference delivered before the second international congress of mathematicians hold at Paris in the 8th point about «the prime numbers problems »: « … and perhaps after an exhaustive discussion of the Riemann formula on prime numbers we will be in a position to reach the rigorous solution of the Goldbach problem i.e.: if any even integer is a sum of two positive prime integers? …» [7].

In 1940, the English Mathematician G.H.Hardy (1877-1947) writed: « it exists some theorems such 'the Goldbach theorem" which did not be proved and which any stupid could conjecture » [6] [10]

In 1977, the American Mathematician (of Polish descent) H.A.Pogorzelski (1922-2015) [8] affirmed to prove the Goldbach conjecture, but his proof is not generally accepted.

In 2000, Faber and Faber devoted $1000000 for any one proving the Goldbach conjecture between March 2000 and March 2002, but no one could give a proof and the question remained open [2][10].

However, the Goldbach conjecture was verified for all the entire even values of the integer $n, 4 \leq n \leq m$, where $m = 10^4$ by Desboves in 1885, $m = 10^5$ by N.Pipping in 1938, $m = 10^8$ by M.L.Stein and P.R.Stein in 1965,

$m = 2.10^{10}$ by A.Granville and J.Van der lune and H.J.J Te Riele in 1989, $m = 4.10^{11}$ by M.K.Sinisalo in 1993, $m = 10^{14}$ by J.M.Deshouillers and H.J.J.Te Riele and Y.Saouter en 1998, $m = 4.10^{14}$ by J.Richstein in 2001, $m = 2.10^{16}$ by T.Oliveira E Silva on 3/14/2003 and $m = 6.10^{16}$ by T.Oliveira E Silva on 10/3/2003 (See[10], [18] and there references). According to [17]: (T. Oliveira e Silva ran a distributed computer search that has verified the conjecture for $n \leq 4 \times 10^{18}$ (and double-checked up to $4 \times 10^{17}$) as of 2013. One record from this search is that 3325581707333960528 is the smallest number that cannot be written as a sum of two primes where one is smaller than 9781)

Finally The Nice University (France) devoted online, since 1999, a sit [11] giving, all the Goldbach decompositions in sums of two prime integers of higher values of even integers.
For more information see [17].

**The note:** my purpose in the present brief note is to show the Goldbach conjecture by using, essentially, the elementary topological properties of continuity satisfied by the integer part function. The main result of the paper is:

**Theorem:** $\forall n \in \mathbb{N}, n \geq 2, \exists (p,q)$ two prime integers such that: $2n = p + q$.

**Methods:** Considering for $n \geq 2$the sets:
$*A_n = \{t \in [p_n, p_{n+1}], \exists p, q \in \mathbb{P}$: such that $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ and $E(t) = \frac{p+q}{2}\}$
$*$and $B_n = \{t \in [p_n, p_{n+1}],\ [p_n, t] \subset A_n\}$,

I show that:$A_n = [p_n, p_{n+1}]$, where $(p_n)_{n\geq 1}$ is the strictly increasing sequence of prime integers.
So: $[2, +\infty[ = \cup_{n=1}^{+\infty} [p_n, p_{n+1}[ \ \Rightarrow \forall n \geq 2 \exists (p,q)$ two prime integers such that$2n = p + q$

**Organization of The paper:** The paper is organized as follows. The §1 is an introduction containing the necessary definition and some History. The §2 contains the ingredients of the proofs. The §3 contains the proof of our main result. The §4 contains the references of the paper given for further reading.

## INGREDIENTS OF THE PROOFS

**Notation:** the closed, the semi-open and the open intervals of $\mathbb{R}$, are (respectively) denoted as below (if $a < b$):
$[a,b]$=$\{t \in \mathbb{R}, a \leq t \leq b\}$,$[a,b[$=$\{t \in \mathbb{R}, a \leq t < b\}$,},$]a,b]$=$\{t \in \mathbb{R}, a < t \leq b\}$, $]a,b[$=$\{t \in \mathbb{R}, a < t < b\}$
Remark that: [a, a] = {a}

**Definition2:** The absolute value function ||is defined on $\mathbb{R}$ by $|t| = \begin{cases} t \ if \ t > 0 \\ 0 \ if \ t = 0 \\ -t \ if \ t < 0 \end{cases}$

**Definition3:** A positive integer $p$ is prime if its set of divisors is $D(p) = \{1, p\}$. The set of all prime integers is denoted $\mathbb{P}$. For $t \geq 2$, we define the set: $\mathbb{P}_t = \{p \in \mathbb{P}, p \leq t\}$ which is a finite set having the cardinal (the number of elements): $card(\mathbb{P}_t) = \pi(t)$ called the prime-counting function.

**Proposition1:** (Euclid [1]) the set $\mathbb{P}$ of prime integers is a strictly increasing infinite sequence $(p_n)_{n\geq 1} = (2,3,5,7,11,13,17, \dots)$

**Proposition2:** We have:
(i)$[2, +\infty[ = \cup_{n=1}^{+\infty} [p_n, p_{n+1}[$
(ii) In particular:$\forall n \in \mathbb{N}, n \geq 2 \exists \varphi(n) \in \mathbb{N}^*$ such that:$n \in [p_{\varphi(n)}, p_{\varphi(n)+1}[$

**Définition4:** ([12]) we note, for$x \in \mathbb{R}$, by $E(x) \in \mathbb{Z}$ the integer part of the real $x$ i.e. the single integer such that: $E(x) \leq x < E(x) + 1$

**Proposition3:** ([12]) we have:
(i)$\forall x \in \mathbb{Z}: E(x) = x$

(ii)$\forall x \in \mathbb{R} - \mathbb{Z} \;\; E(-x) = -E(x) - 1$
(ii) $0 \leq x < 1 \Rightarrow E(x) = 0$
(iii)$\forall x \in \mathbb{R}: 0 \leq x - E(x) < 1$
(iv) $\forall x, y \in \mathbb{R} \; E(x + y) = E(x) + E(y) + \chi_{[1,2[}(x - E(x) + y - E(y))$
Where: $\chi_{[1,2[}(t) = \begin{cases} 1 \; if \; t \in [1,2[ \\ 0 \; if \; t \notin [1,2[ \end{cases}$ is the characteristic function of the interval [1,2[
In particular:$\forall x \in \mathbb{Z} \forall y \in \mathbb{R} \; E(x + y) = x + E(y)$

**Example:** $E\left(x + \frac{1}{2}\right) = E(x) + \chi_{[1,2[}\left(\frac{1}{2} + x - E(x)\right) = E(x) or \; E(x) + 1$
(v)$\forall x, y \in \mathbb{R} \;\; x < y \Rightarrow E(x) \leq E(y)$

**Definition5:** If $A$ is any subset of any set $X$, we define $A^c = \{x \in X, x \notin A\}$ called the complementary set of $A$.

**Definition 6:** ([13]) (1) A topological space $X$ is a set equipped with a part :$\tau \subset P(X)$ (the set of its parts) called topology such that:
(i)$X, \emptyset \in \tau$
(ii) Any arbitrary (finite or infinite) union of members of $\tau$ (i.e. open subsets) still belongs to $\tau$ $(i.e \; is \; open)$
(iii) The intersection of any finite number of members of $\tau$ ($i.e.$ open subsets)still belongs to $\tau$ (I.e. is open)
(2)An element $U$ of $\tau$ is called an open subset of $X$
(3)For $U \in \tau$: $U^c$ is called a closed subset of $X$

**Proposition4:** ([13]) in ($\mathbb{R}$, ||) (and generally in a metrical space): $U$ open $\Leftrightarrow$ $\forall x \in U \exists \epsilon(x) > 0$ such that : $[x - \epsilon(x), x + \epsilon(x)] \subset U$

**Proposition 5:** ([13]) we have:
(i) Any arbitrary (finite or infinite) intersection of closed subsets of X is still closed.
(ii) The union of any finite number of closed subsets is still closed.

**Definition7:** ([14]) we call adherence of a subset $A$ of a topological space $X$, noted $\bar{A}$ the set:

$$\bar{A} = \bigcap_{all \; the \; closed \; subsets \; of \; X \supset A} F$$

If $X$ is a metrical space $\bar{A} = \{a \in X, \exists a_n \in A: a = \lim_{n \to +\infty} a_n\}$
If Y is a subspace of X (equipped with the induced topology) and $A \subset Y$, then:
The adherence of A relatively to Y is =$Y \cap \bar{A}$

**Example:** if $[a, b] \subset [c, d]$, the adherence of [a, b [relatively to [c, d] is = [a, b]

**Proposition 6:** ([14]) (i) $A \subset \bar{A}$(ii) $\bar{X} = X, \bar{\emptyset} = \emptyset$(iii)$\overline{\cup_{k=1}^{m} A_k} = \cup_{k=1}^{m} \overline{A_k}$ (iv) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$
(v)$A$ Closed$\Leftrightarrow$ $\bar{A} = A$

**Definition8:** ([15]) A function $f: X \to Y$ between two metrical spaces $X, Y$ is continuous in t ∈ $X$ ⇔ $(\forall (t_n)_n \subset X: (\lim_{n \to +\infty} t_n = t) \Rightarrow (\lim_{n \to +\infty} f(t_n) = f(t)))$

**Proposition 7:** the function integer part $E$ is continuous on the set: $\mathbb{R} - \mathbb{Z}$ (the complementary in $\mathbb{R} \; of \; \mathbb{Z}$)

**Proposition8:** ([16]) any real non empty subset bounded by above $A$ has a supremum: sup $(A) \in \bar{A}$. sup $(A)$ is the smallest above bound.

**Proposition9:** (negation of a proposition [18]) the negation of a proposition (P), denoted non (P), is the proposition true when (P) is false and false when (P) is true. We have: non (non (P)) = (P)

**Example:** non $(\forall) = \exists$, non $(\exists) = \forall$, non (=) =≠, non (<) =≥

## PROOF OT THE BINARY GOLDBACH CONJECTURE

**Theorem:** $\forall n$ integer $\geq 2$ $\exists (p,q)$ prime integers such that: $2n = p + q$

**Proof:** (of the theorem)
The Proof of the theorem will be deduced from the claims below.

**Definition9:** For $n \in \mathbb{N}, n \geq 2$, let
* $A_n = \{t \in [p_n, p_{n+1}], \exists p,q \in \mathbb{P}$ such that: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ *and* $E(t) = E\left(t+\frac{1}{2}\right) = \frac{p+q}{2}\}$
*$B_n = \{t \in [p_n, p_{n+1}],\ [p_n, t] \subset A_n\}$ And $C_n = A_n^c$

**Claim1:** We have:

$$C_n = \{t \in [p_n, p_{n+1}] \text{ such that: } \forall p,q \in \mathbb{P}: 3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1} \begin{cases} \mathrm{E}\left(t+\frac{1}{2}\right) = \mathrm{E}(t) + 1 \\ \text{or } |\mathrm{E}\left(t+\frac{1}{2}\right) - \frac{p+q}{2}| \geq 1\} \\ \text{or } |\mathrm{E}(t) - \frac{p+q}{2}| \geq 1 \end{cases}$$

**Proof:** (of claim1)
*The result is evident by taking the negation of the relation defining the set $A_n$
*Indeed, we have:
$t \in C_n \Leftrightarrow non(\exists p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$such that $E(t) = E\left(t+\frac{1}{2}\right) = \frac{p+q}{2})$
$\Leftrightarrow$
$non(\exists p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$such that $E(t) = E\left(t+\frac{1}{2}\right)$ and $E(t) = \frac{p+q}{2}$ and $E\left(t+\frac{1}{2}\right) = \frac{p+q}{2})$
$\Leftrightarrow non(\ E(t) = E\left(t+\frac{1}{2}\right)$ and $\exists p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$such that $E(t) = \frac{p+q}{2}$
and $E\left(t+\frac{1}{2}\right) = \frac{p+q}{2})$
$\Leftrightarrow (\ E(t) \neq E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ $E(t) \neq \frac{p+q}{2}$ or $E\left(t+\frac{1}{2}\right) \neq \frac{p+q}{2})$
$\Leftrightarrow (\ E(t) + 1 = E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ $\left|E(t) - \frac{p+q}{2}\right| > 0$ or $\left|E\left(t+\frac{1}{2}\right) - \frac{p+q}{2}\right| > 0)$
$\Leftrightarrow (\ E(t) + 1 = E\left(t+\frac{1}{2}\right)$ or $\forall p,q \in \mathbb{P}$ satisfying: $3 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ $\left|E(t) - \frac{p+q}{2}\right| \geq 1$ or $\left|E\left(t+\frac{1}{2}\right) - \frac{p+q}{2}\right| \geq 1)$

**Claim2:** We have: $p_n \in A_n$ and $p_n \in B_n$, so: $A_n \neq \emptyset$ *and* $B_n \neq \emptyset$

**Proof:** (of claim2)
For $n \geq 2:\ 2E(p_n) = 2E(p_n + \frac{1}{2}) = 2p_n = p_n + p_n$, with $p_n \in \mathbb{P}, 3 = p_2 \leq p_n < p_{n+1} \Rightarrow p_n \in A_n \Rightarrow \{p_n\} = [p_n, p_n] \subset A_n \Rightarrow p_n \in B_n$

**Claim3:** $A_n$ is closed in $[p_n, p_{n+1}]$

**Proof:** (of claim3)
*Let $(t_m)_m \subset A_n$ converging to $t \in [p_n, p_{n+1}]$, show that $t \in A_n$
*We have:
**$(t_m)_m \subset A_n \Rightarrow E(t_m) = E\left(t_m + \frac{1}{2}\right) = \frac{x_m + y_m}{2}$ with $x_m, y_m \in \mathbb{P}: 3 \leq p_n \leq \frac{x_m+y_m}{2} \leq p_{n+1}$
**$x_m, y_m \in \mathbb{P}$ being bounded by 3 and $p_{n+1}$, we have:
$\exists N \in \mathbb{N} \exists p,q \in \mathbb{P}\ 2 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ Such that: $x_m = p$ *and* $y_m = q \forall m \geq N$

**First case:** if $t \notin \mathbb{N}$ (so the integer part function$E$ is continuous in $t$)

$$\begin{cases} E(t_m) = \frac{p+q}{2} \ \ \forall m \geq N \\ \lim_{m\to+\infty} t_m = t \notin \mathbb{N} \\ E \text{ continuous in } t \end{cases} \Rightarrow \frac{p+q}{2} = \lim_{m\to+\infty} E(t_m) = E(\lim_{m\to+\infty} t_m) = E(t)$$

**Second case:** if $t \in \mathbb{N}$ (so $E$ is continuous in: $t + \frac{1}{2} \notin \mathbb{N}$)

$$\begin{cases} E\left(t_m + \frac{1}{2}\right) = \frac{p+q}{2} \ \ \forall m \geq N \\ \lim_{m\to+\infty} t_m = t \in \mathbb{N} \\ E \text{ continuous in } t + \frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{p+q}{2} = \lim_{m\to+\infty} E\left(t_m + \frac{1}{2}\right) = E(\lim_{m\to+\infty} t_m + \frac{1}{2}) = E(t + \frac{1}{2})$$

*So, we have: $\exists p, q \in \mathbb{P}\ 2 \leq p_n \leq \frac{p+q}{2} \leq p_{n+1}$ such that $E(t) = E\left(t + \frac{1}{2}\right) = \frac{p+q}{2}$

*So, the claim3 is then showed.

**Claim4:** $B_n$ has a supremum $\sup(B_n) = \alpha(n)$

**Proof:** (of claim 4)

*By definition of $B_n$ this set is bounded above (by $p_{n+1}$) and by claim 2 is non empty

*So: the result follows by combination of proposition8 and claim2.

**Claim5:** We have: $[p_n, \alpha(n)[\subset B_n$

**Proof:** (of claim5)

*Let $l \in [p_n, \alpha(n)[$

*By definition of $\alpha(n) = \sup(B_n)$ : $\exists x \in B_n$ such that: $l \leq x$

*Indeed, if not, we have: $\forall x \in B_n\ l > x$, i.e. $l$ is an above bound of $B_n$

*So:$\alpha(n)$, being by proposition 3, the smallest above bound, we have: $l \geq \alpha(n)$

*This contradicting our hypothesis"$l \in [p_n, \alpha(n)[$", the result follows.

**Claim6:** We have: $[p_n, \alpha(n)[\subset A_n$

**Proof:** (of claim 6)

*Let $l \in [p_n, \alpha(n)[$

*By claim 5: $l \in B_n$

*So, by definition of $B_n$, $[p_n, l] \subset A_n$

*In particular: $l \in A_n$

*That is: $[p_n, \alpha(n)[\subset A_n$

*This ends the proof of claim 5.

**Claim7:** we have: $[p_n, \alpha(n)] \subset A_n$

**Proof:** (of claim 7)

*By claim 6, we have: $[p_n, \alpha(n)[\subset A_n$

*But, by of claim 3, $A_n$ is closed.

*So, by the example following definition 7 and the assertion (v) of proposition6, $\overline{A_n} = A_n \Rightarrow \overline{[p_n, \alpha(n)[} = [p_n, \alpha(n)] \subset \overline{A_n} = A_n$

*The result follows.

**Claim 8:** We have: $B_n = [p_n, \alpha(n)]$

**Proof:** (of claim 8)

*By combination of claim5 and claim7, we have: $[p_n, \alpha(n)] \subset B_n$

*But, by definition of $\alpha(n)$, $l \in B_n \Rightarrow p_n \leq l \leq \alpha(n)$

*That is: $B_n \subset [p_n, \alpha(n)]$

*The result follows.

**Claim9:** If $\alpha(n) < p_{n+1}$, then $\exists h \in ]0, p_{n+1} - \alpha(n)]$ such that: $[\alpha(n), \alpha(n) + h] \subset A_n$

**Proof:** (of claim 9)

*Suppose contrarily that: $\forall h > 0\ [\alpha(n), \alpha(n) + h]$ is not contained in $A_n$

*That is: $\forall h > 0 \exists x(h) \in [\alpha(n), \alpha(n)+h]$, such that $x(h) \notin A_n$ i.e. $x(h) \in C_n = A_n^c$
*So: $\forall h > 0 \exists y(h) \in ]0, h]$ such that: $x(h) = \alpha(n) + y(h) \in C_n$
*By definition of $C_n$ (according to claim 1)) we have:
$E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E(\alpha(n) + y(h)) + 1$ *or* $\forall p, q \in \mathbb{P}\ 3 \le p_n \le \frac{p+q}{2} \le p_{n+1}\ |E(\alpha(n) + y(h)) - \frac{p+q}{2}| \ge 1$ *or* $|E(\alpha(n) + y(h) + \frac{1}{2}) - \frac{p+q}{2}| \ge 1$

**<u>First case:</u>** $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E(\alpha(n) + y(h)) + 1$

**<u>First under-case:</u>** if $\alpha(n) \in \mathbb{N}$
*We have: $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = \alpha(n) + E\left(y(h) + \frac{1}{2}\right) = \alpha(n) + 0 = \alpha(n) = E(\alpha(n) + y(h)) + 1 = \alpha(n) + E(y(h)) + 1 = \alpha(n) + 0 + 1 = \alpha(n) + 1$

*This being impossible: the first under case cannot occur.

**<u>Second under-case:</u>** if $\alpha(n) \notin \mathbb{N}$ (so the function $E$ is continuous in $\alpha(n)$)

**<u>The first case under the second under-case:</u>** if $\alpha(n) + \frac{1}{2} \in \mathbb{N}$
$* E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = \alpha(n) + \frac{1}{2} + E(y(h)) = \alpha(n) + \frac{1}{2} + 0 = E(\alpha(n) + \frac{1}{2}) = E(\alpha(n) + y(h)) + 1 = E(\alpha(n)) + 1$
*This contradicting claim7 assuring that: $\alpha(n) \in A_n$ (i.e.: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))$) this case cannot occur.

**<u>The second case under the second under-case:</u>** if $\alpha(n) + \frac{1}{2} \notin \mathbb{N}$ (so the function $E$ is continuous in $\alpha(n) + \frac{1}{2}$)
*By tending: $h \to 0$ (noting that, then: $y(h) \to 0$)in the following relation:

$$E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E(\alpha(n) + y(h)) + 1$$

we have: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))) + 1$
*This contradicting claim7 assuring that: $\alpha(n) \in A_n$ (i.e.: $E\left(\alpha(n) + \frac{1}{2}\right) = E(\alpha(n))$) this case cannot occur.
*So the second under-case cannot, also, occur, because the two possible under-cases are impossible.

**<u>Conclusion:</u>** the first-case cannot occur because the two possible under-cases are impossible

**<u>Second case:</u>** $E\left(\alpha(n) + y(h) + \frac{1}{2}\right) = E(\alpha(n) + y(h))$
We have: $\forall p, q \in \mathbb{P}\ 3 \le p_n \le \frac{p+q}{2} \le p_{n+1}\ |E(\alpha(n) + y(h)) - \frac{p+q}{2}| = |E(\alpha(n) + y(h) + \frac{1}{2}) - \frac{p+q}{2}| \ge 1$

**<u>First under- case:</u>** if $\alpha(n) \notin \mathbb{N}$ (so: $E$ is continuous in $\alpha(n)$)
*By the assertion (i) of proposition 7: $f(t) = E(t)$ is continuous on $\alpha(n)$, so:
$\lim_{h\to 0} y(h) = 0 \Rightarrow \forall p, q \in \mathbb{P}\ 3 \le p_n \le \frac{p+q}{2} \le p_{n+1} \lim_{h\to 0} |E(\alpha(n) + y(h)) - \frac{p+q}{2}| = |E(\alpha(n)) - \frac{p+q}{2}| \ge 1$
*This contradicting claim 7 (assuring that: $\exists p, q \in \mathbb{P}$ such that $3 \le p_n \le \frac{p+q}{2} \le p_{n+1}$and $E(\alpha(n)) = \frac{p+q}{2}$), this case cannot occur.

**<u>Second under- case:</u>** if $\alpha(n) \in \mathbb{N}$, we have:
$\forall p, q \in \mathbb{P}\ 3 \le p_n \le \frac{p+q}{2} \le p_{n+1}\ |E(\alpha(n) + y(h)) - \frac{p+q}{2}| = \left|\alpha(n) + E(y(n)) - \frac{p+q}{2}\right| = |E(\alpha(n)) - \frac{p+q}{2}| \ge 1$
*This contradicting claim 7 (assuring that: $\exists p, q \in \mathbb{P}$ such that $3 \le p_n \le \frac{p+q}{2} \le p_{n+1}$and $E(\alpha(n)) = \frac{p+q}{2}$), this case cannot occur.

*This being impossible, the second case cannot, also, occur.

**Conclusion:** The two possible cases coulding not both occur, our starting absurd hypothesis "$\forall h > 0$ $[m, m+h]$ is not contained in $A_n$" is not true, so its negation:" $\exists h \in ]0, p_{n+1} - \alpha(n)]$ such that: $[\alpha(n), \alpha(n)+h] \subset A_n$" is true.

**Claim10:** $\alpha(n) = p_{n+1}$
**Proof:** (of claim10)
*Suppose contrarily that:$\alpha(n) < p_{n+1}$.
*By claim 9: $\exists h \in ]0, p_{n+1} - \alpha(n)]$such that: $[\alpha(n), \alpha(n)+h] \subset A_n$
*So, by claim7 and claim 9, we have:
$[p_n, \alpha(n)] \subset A_n$ and $[\alpha(n), \alpha(n)+h] \subset A_n \Rightarrow [p_n, \alpha(n)+h] = [p_n, \alpha(n)] \cup [\alpha(n), \alpha(n)+h] \subset A_n$
*That is, by definition of $B_n$, $\alpha(n)+h \in B_n$
*But, by claim 8: $B_n = [p_n, \alpha(n)]$
*So : $\alpha(n)+h \in [p_n, \alpha(n)]$ for $h > 0$ is impossible.

**Conclusion:** That is our absurd starting hypothesis «$\alpha(n) < p_{n+1}$ » is false and its negation «$\alpha(n) = p_{n+1}$. » is true.

**RETURN TO THE PROOF OF THE THEOREM**
*By combination of claim8 and claim10, we have:$\forall n$ integer $\geq 2$ $[p_n, p_{n+1}] = B_n$
*But by the assertion (ii) of proposition2:

$$\forall n \text{ integer} \geq 3 \; \exists \varphi(n) \in \mathbb{N}^*, \varphi(n) \geq 2 \text{ such that: } n \in \left[p_{\varphi(n)}, p_{\varphi(n)+1}\right] = B_{\varphi(n)}$$

*Having $2 \times 2 = 2 + 2$ (with 2 prime) we have, by definition of $B_{\varphi(n)}$:

$$\forall n \geq 2 \; \exists (p, q) \text{ Two prime integers such that:} 2n = p + q$$

*This ends the proof of the Goldbach conjecture

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