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CONFIRMATION OF THE IRRATIONALITY OF THE EULER CONSTANT

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ABSTRACT

I prove that the Euler constant is irrational by showing that it cannot be written as a quotient of two integers. In fact, going from the rational form $\frac{p}{q}$ of the Euler constant, we arrive to the contradiction that $\sum_{k=1}^{n}(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right))$ is rational for some n>2q.

CITATION

"A mathematical theory is not be considered complete until you have made it so clear that you can explain it to the first man you meet on the street"
A French mathematician
Cited by D.Hilbert in his lecture delivered to the international
Congress of mathematicians
at Paris in 1900

RÉSUMÉ

Je montre que la constante d'Euler, bien connue de l'analyse élémentaire, est irrationnelle, en montant qu'elle ne peut s'écrire comme quotient de deux entiers. En effet partant de la forme rationnelle $\frac{p}{q}$ de la constante d'Euler, j'arrive à la contradiction que $\sum_{k=1}^{n}(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right))$ est rationnel pour uncertain entier $n > 2q$.

KEYWORDS: Euler constant; Irrationality; Integer part; 2-adic development 2010 Mathematics Subject Classification/ 11 A xx (Elementary Number Theory).

INTRODUCTION

Definition1: *we call Euler constant or also Euler-Mascheroni constant, the real number denoted by γ or C such that:*

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n}\frac{1}{k} - \ln(n)$$

We show [12] that the first 32 decimals of γ are:

γ=0.5772 1566 4901 5328 6060 6512 0900 8240…..

History: this constant was introduced by Leonhard Euler (1707-1783) in 1734 when he determined the five first decimals of the constant and denoted it C [2].
In1736, he determined 15 decimals by using the Euler-Maclaurin summation [3].

In 1790, Lorenzo Mascheroni (1750-1800) determined 32 decimals (of which only 19 are exact) and contributed to the knowledge of the constant and proposed to note it γ [11].

Between 1809 and 1887 the determination of decimals of γ was continued by many Mathematicians [5].

David Hilbert (1862-1943) declared that the irrationality of the Euler constant seems to be an unapproachable problem" in front of which mathematicians stand helpless" [6][14].

Godefry Harold Hardy (1877-1947) was perplexed by the properties of γ to the point that he has declared a day that he will give his chair at Oxford to the person that can prove the irrationality of this constant [1][6][14].

The determination of decimals of this constant was retaken in 1952 and passed from 1271 decimals in 1962 by D.E. Knuth to 108.000.000 decimals in 1999 by X.Gourdon and P.Demichel [5]. The world record of 2 billion digits is due to S.Kondo [8] [14].

The note: from 1734 to now we do not know if γ is irrational, rational, algebraic or transcendental.
The purpose of this short note is to prove, using relatively elementary tools, that γ is irrational by showing that it cannot be written as a quotient of two integers. In fact, going from the form $\frac{p}{q}$ (p and q being two natural integers) of Euler constant, we arrive to the contradiction "$q\sum_{k=1}^{n}\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)$ *is rational*" for some integer n>2q.

The note is organized as follows. After an introduction giving the definition and some history, the second paragraph, entitled "Ingredients of the proof", contains the results used in the proof of our main result for which we have devoted a third paragraph.

INGREDIENTS OF THE PROOF:

We will need, for the proof of our result, the following facts:
Proposition1 *(Hermite-Lindemann theorem) [4][7][9][10][13]*
For any, not null, algebraic number a, the number e^a is transcendental.
Recall that a real number is algebraic if it is a root of a polynomial with rational coefficients, and that it is transcendental in the opposite case.
In particular, a rational number is algebraic, and a transcendental number is necessarily irrational.
Corollary1: *for any, not null, rational number r: e^r is irrational.*
Corollary2*: for any, not null, natural integer n: ln (n+1) is irrational.*
Where ln(x) denotes the Neperian logarithm
Proposition2: $\forall t\geq 1 \quad \frac{1}{t+1}\leq \ln(1+\frac{1}{t})\leq \frac{1}{t}$.
Proof :(Of proposition2)
°the first inequality comes from the fact that: if we let $g(t)=\frac{1}{t+1}-\ln(1+\frac{1}{t})$, we have:

$$g'(x)=\frac{1}{t(t+1)^2}>0. \text{ So: } g(t)\leq \lim_{y\to+\infty} g(y)=0 \text{ for any } t\geq 1.$$

°the second inequality comes from the fact that if we let $h(t)=\frac{1}{t}-\ln(1+\frac{1}{t})$, then one has:

$$h'(t)=-\frac{1}{t^2(t+1)}<0. \text{ So: } h(t)\geq \lim_{y\to+\infty} h(y)=0 \text{ for any } t\geq 1.$$

Definition2: we note by [t] the integer part of the real t. By definition, it is the single integer m such that $m\leq t<m+1$. We will need the three following properties of the integer part:
Proposition3: *We have:*$\forall t\in\mathbb{R}\backslash\mathbb{Z}\quad [-t]=-[t]-1$
Proposition4: We have:

$$\forall t,s\in\mathbb{R}\quad [s+t]=[s]+[t]+\chi_{[1,+\infty[}(s-[s]+t-[t])$$

Where χ_A denotes the characteristic function of the set A defined by:

$$\chi_A(t)=\begin{cases}0 & if\ t\notin A\\ 1 & if\ t\in A\end{cases}$$

Proposition5: *for any real t, and for any natural integer s≥2, we have:*

$$[st]=\sum_{i=0}^{s-1}[t+\tfrac{i}{s}]=s[t]+\sum_{i=1}^{s-1}\chi_{[1,+\infty[}(t-[t]+\tfrac{i}{s})$$

Proof :(Of proposition5)
*The first equality is obtained by recurrence on s.
*The second is obtained by using proposition4 above.
Proposition6: *for any real t, one has:* $\lim_{N\to+\infty}\frac{[Nt]}{N}=t$.
Proof :(Of proposition6)
We have: $[Nt]\leq Nt<[Nt]+1$ and $\frac{[Nt]}{N}\leq t<\frac{[Nt]}{N}+\frac{1}{N}$.
So the result follows by tending N to infinity.
Proposition7: *for any real number t, we have:*

$$t-[t]=\sum_{m=1}^{+\infty}\frac{[2^m t]-2[2^{m-1}t]}{2^m}$$

Proof: (Of proposition7)
If we let:$u_m=\frac{[2^m t]}{2^m}$, we have, by proposition6:

$\sum_{m=1}^{+\infty}\frac{[2^m t]-2[2^{m-1}t]}{2^m}=\sum_{m=1}^{+\infty}(u_m - u_{m-1})=\lim_{m\to+\infty} u_m - u_0 = t-[t]$

Proposition8 : *(terms grouping rule)*

1) If u_n is the general term of a series such that $\lim_{n\to+\infty} u_n = 0$,

2) if k is a natural integer,

And if

3)$\varphi: \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping such that:$\varphi(m+1) - \varphi(m) \le k$, for any natural integer m,

Then:

$$\sum_{m=m_0}^{+\infty} u_m = \sum_{m=m_0}^{+\infty} \sum_{j=\varphi(m)}^{\varphi(m+1)-1} u_j$$

THE IRRATIONALITY OF EULER CONSTANT:

Theorem: *the Euler constant is irrational.*

Proof: (Of the theorem)

*suppose that: $\gamma = \frac{p}{q}$ with p and q two, not null, natural integers.

Lemma1: *for any not null natural integer, we have*

$$q\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) \le p \le q\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) + \frac{q}{n+1}$$

Proof :(Of lemma1)

*the sequence $u_n = \left(\sum_{k=1}^{n}\frac{1}{k}\right) - \ln(1+n) = \sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)$is increasing positive by proposition2.

With $\gamma = \lim_{n\to+\infty} u_n \ge u_m$ for any natural integer m not null.

*the sequence $v_n = \left(\sum_{k=1}^{n+1}\frac{1}{k}\right) - \ln(1+n) = u_n + \frac{1}{n+1}$ is decreasing positive by proposition2.

(Because $v_n - v_{n-1} = \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) < 0$), with $\gamma = \lim_{n\to+\infty} v_n \le v_m$, for any natural integer m not null.

*the result follows.

Lemma2*: for any not null natural integer n:*

$$\left(\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)\right) = \sum_{k=1}^{n}\frac{1}{k} - \ln(n+1) \text{ is irrational.}$$

Proof: (Of lemma2)

The result follows from corollary2.

Lemma3: *for any not null natural integers n and s such that: n>sq, we have:*

$$\left[sq\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)\right] = sp - 1$$

Proof :(Of lemma3)

By lemma1 and lemma2, we have: for n>sq

$$sp > sq\sum_{k=1}^{n}\left(\frac{1}{k} - ln\left(1+\frac{1}{k}\right)\right) \ge sp - \frac{sq}{n+1} \ge sp \qquad +\left[-\frac{sq}{n}\right] = sp - 1$$

The result follows.

Lemma4: *if n is a natural integer such that n>2q and r= $\left[\frac{\ln(n)-\ln(q)}{\ln(2)}\right]$, then r satisfies the property: $q2^r<n$, so (by lemma3) for any integer m such that $1\le m\le r$ (n>2q assuring that r ≥1) one has:*

$$\left[2^m q\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)\right] = 2^m p - 1$$

Proof: (Of lemma4)

It is evident that $2^r \le 2^N = \frac{n}{q}$ (if $N = \frac{\ln(n)-\ln(q)}{\ln(2)}$).

So, the result follows from lemma3.

Lemma5: *for any not null natural integer n such that n>2q, we have:*

$$m \le r \Rightarrow \forall 1 \le k \le m: \quad [2^k x] - 2[2^{k-1}x] = 1$$

(Where: $r = \left[\frac{\ln(n)-\ln(q)}{\ln(2)}\right]$ and $x = q\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)$)

Proof: (Of lemma5)

*By lemma4, we have:

$$[2^m x] = 2^m p - 1 \text{ And } [2^{m-1}] = 2^{m-1}p - 1$$

*So:$[2^m x] - 2[2^{m-1}x] = (2^m p - 1) - 2(2^{m-1}p - 1) = 1$.

The proof is finished.

Lemma6: *We have:*

$$[2^m x] = 2^m p - 1 \Rightarrow m \le r.$$

(Where $x = q\sum_{k=1}^{n}(\frac{1}{k} - \ln(1+\frac{1}{k}))$ and $r = [\frac{\ln(n)-\ln(q)}{\ln(2)}]$ for $n > 2q$ a very great prime integer)

Proof :(Of lemma6)

*Suppose that:$[2^m x] = 2^m p - 1$

*Show by recurrence on m that: $m \le r (\Leftrightarrow 2^m \frac{q}{n} < 1 \Leftrightarrow \left[2^m \frac{q}{n}\right] = 0)$.

*For: $m = 1$, one has: $m = 1 \le r$

*Suppose: $2^{m-1}\frac{q}{n} < 1$ (So: $2^m \frac{q}{n} < 2$).

*But we have, by lemma1: $p - \frac{q}{n+1} < x < p$.

*So, we have:

$[2^m x] = 2^m p - 1 \Rightarrow 2^m p - 2 < 2^m p - 2^m \frac{q}{n} - 1 < 2^m p - \left[2^m \frac{q}{n}\right] - 1 = 2^m p + [-\frac{2^m q}{n}]$

$\le \left[2^m\left(p - \frac{q}{n+1}\right)\right] < 2^m x < 2^m p$

Remark:

*By the recurrence hypothesis, we have: 0<$2^m \frac{q}{n} < 2$.

*So:$2^m \frac{q}{n} \in \mathbb{Z} \Leftrightarrow 2^m \frac{q}{n} = 1 \Leftrightarrow n = 2^m q$.

*Then choosing, for example, n as a prime integer (Which is necessarily ≥ 3 because $n > 2q > 2$) assures that:

$[-2^m \frac{q}{n}] = -\left[2^m \frac{q}{n}\right] - 1$

Recall that a prime integer s has its set of divisors= $\{1, s\}$.

*Finally we have necessarily:

$$[2^m x] = 2^m p - 1 = 2^m p - \left[2^m \frac{q}{n}\right] - 1 \Rightarrow \left[2^m \frac{q}{n}\right] = 0 \Leftrightarrow 2^m \frac{q}{n} < 1$$

Conclusion: This finishes the recurrence of lemma6.

Lemma7: *We have:* $\sum_{m=r+1}^{+\infty} \frac{[2^m x] - 2[2^{m-1}x]}{2^m} = \sum_{m=r+1}^{+\infty} \sum_{k=mr}^{mr+r-1} \frac{[2^k x] - 2[2^{k-1}x]}{2^k}$

(Where: $x = q\sum_{k=1}^{n}(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right))$ and $r = \left[\frac{\ln(n)-\ln(q)}{\ln(2)}\right]$ $for\ n > 2q$ *)*

Proof: (Of lemma7)

The proof follows by application of proposition8, by taking: $\varphi(m) = rm$

Lemma8: *Let an integer* $m \ge r + 1$, *we have:*

$$mr \le k \le mr + r - 1 \Rightarrow [2^k x] - 2[2^{k-1}x] = 0$$

(x=$q\sum_{k=1}^{n}(\frac{1}{k} - \ln(1+\frac{1}{k}))$ *and:* r=$\left[\frac{\ln(n)-\ln(q)}{\ln(2)}\right]$ $for\ n > 2q\ a\ very\ great\ prime\ integer$)

Proof: (Of lemma8)

Remark: Recall that by proposition5, we have:

$$[2^k x] - 2[2^{k-1}x] = \chi_{[1,+\infty[}\left(2^{k-1}x - [2^{k-1}x] + \frac{1}{2}\right) \in \{0,1\}$$

*Suppose that, for certain $k \in [mr, mr + r - 1]$, we have:$[2^k x] - 2[2^{k-1}x] = 1$

Claim1: *for any j such that:*$mr \le j \le k$, *we have:*$[2^j x] - 2[2^{j-1}x] = 1$

Proof :(Of Cleam1)

*Do a recurrence on j.

*The proposition is true for:$j = k$.

*Suppose that for any $i \in [j, k]$, one has:$[2^i x] - 2[2^{i-1}x] = 1$, and show that:

$$[2^{j-1}x] - 2[2^{j-2}x] = 1$$

Claim2: *We have* $[2^k x] = 2^j[2^{k-j}x] + 2^j - 1$

Proof :(Of cleam2)

*The proof follows from the recurrence hypothesis ($i.e$: $\forall i \in [j,k]$ $[2^i x] = 2[2^{i-1}x] + 1$)

*Indeed:

$[2^k x] = 2[2^{k-1}x] + 1$ =$2^2[2^{k-2}x] + 2^2 - 1 = \cdots = 2^j[2^{k-j}x] + 2^j - 1$.

Claim3: *We have:* $[2^k x] = 2^k p - 1$*(So:* $k \le r$*)*

Proof: (Of cleam3)

*Because: j$\ge mr$, we have necessarily:$k - j \le r - 1$

*Indeed: $k - j > r - 1$ and $j \ge mr \Rightarrow mr + r - 1 \ge k = j + k - j > mr + r - 1$

*This contradiction shows the affirmation.

*Now, by lemma4: $k - j \le r - 1 \Rightarrow [2^{k-j}x] = 2^{k-j}p - 1$

*So, Cleam2⇒ $[2^k x] = 2^k p - 1$

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*Then: lemma6⇒ $k \leq r$

Return to the proof of claim1:

*In particular: lemma5⇒ $[2^{j-1}x] = 2[2^{j-2}] + 1$

*So the recurrence is finished.

Return to the proof of lemma8:

*Now: cleam3⇒ $mr \leq k \leq r \Rightarrow m \leq 1$.

*This contradicting our choice: $m \geq r + 1$, we have:

$$\forall k \in [mr, mr + r - 1] \quad [2^k x] - 2[2^{k-1}x] = 0.$$

*The proof of lemma8 is finished.

***Lemma9 :(the wanted contradiction)** for a very great prime integer not null n such that n>2q, we have:*

$$q\sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right) = [q\sum_{k=1}^{n}(\frac{1}{k} - \ln(1 + \frac{1}{k}))] + 1 - 2^{-r} \in \mathbb{Q}$$

(Where: $r = [\frac{\ln(n) - \ln(q)}{\ln(2)}]$)

Proof :(Of lemma9)

Let: $x = q\sum_{k=1}^{n}(\frac{1}{k} - \ln(1 + \frac{1}{k}))$

Claim4: $\sum_{m=1}^{r} 2^{-m}([2^m x] - 2[2^{m-1}x]) = 1 - \frac{1}{2^r}$

Proof: (Of claim4)

By lemma5: $1 \leq m \leq r \Rightarrow [2^m x] - 2[2^{m-1}x] = 1$.

So: $\sum_{m=1}^{r} 2^{-m}([2^m x] - 2[2^{m-1}x]) = \sum_{m=1}^{r} 2^{-m} = 1 - 2^{-r}$

Claim5: $\sum_{m=r+1}^{+\infty}\frac{[2^m x] - 2[2^{m-1}x]}{2^m} = 0$

Proof: (Of claim5)

By lemma7 and lemma8, we have:

$\sum_{m=r+1}^{+\infty}\frac{[2^m x] - 2[2^{m-1}x]}{2^m} = \sum_{m=r+1}^{+\infty}\sum_{k=mr}^{mr+r-1}\frac{[2^k x] - 2[2^{k-1}x]}{2^k} = 0.$

Return to the proof of lemma9:

We have, by using proposition7, claim4 and claim5:

$x - [x] = \sum_{m=1}^{+\infty}\frac{[2^m x] - 2[2^{m-1}x]}{2^m}$

$= \sum_{m=1}^{r}\frac{[2^m x] - 2[2^{m-1}x]}{2^m} + \sum_{m=r+1}^{+\infty}\frac{[2^m x] - 2[2^{m-1}x]}{2^m}$

$= 1 - \frac{1}{2^r} + 0 = 1 - \frac{1}{2^r}$

The result follows

CONCLUSION

By combination of lemma2 and lemma9, we get the contradiction that:

$$\sum_{k=1}^{n}\frac{1}{k} - \ln(n + 1) \text{ is rational for some n.}$$

So the Euler constant cannot be written as a quotient of two integers, that it is irrational. This ends the proof of the theorem.

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