GLOBAL JOURNAL OF ADVANCED ENGINEERING TECHNOLOGIES AND SCIENCES

CONFIRMATION OF THE RIEMANN HYPOTHESIS

Mohammed Ghanim*
* Ecole Nationale de Commerce et de Gestion, Tanger, B.P 1255, Maroc

Citation
« Le génie c'est la simplicité »
The French mathematician René Descartes (1596-1650)-Œuvres complètes

ABSTRACT

I show, in this original research paper, that the famous Riemann Hypothesis is true by proving that its simple Balazard equivalent form (cited, without proof, in [1] p 16 as the equivalent form 5), which I prove here using [22] and [30], is true. The proof is essentially based on elementary tools of mathematics; it uses the topological properties of the functions:

$$s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi} \text{ and } s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$$

RÉSUMÉ

Je montre que la fameuse hypothèse de Riemann est vraie en montrant que sa forme équivalente simple de Balazard (citée, sans preuve, dans [1] p 16, comme la forme équivalente 5) est vraie en utilisant [22] et [30]. La démonstration est fondée, essentiellement, sur des outils de mathématiques de base en exploitant les propriétés topologiques des fonctions :

$$s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi} \quad et\ s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$$

KEYWORDS: Riemann zeta function; Riemann hypothesis; Koch equivalent form of the Riemann hypothesis (as sharpened by Schoenfeld); Upper semi continuity.

2010 Mathematics Subject Classification: 11 A xx (Elementary Number Theory)

INTRODUCTION

Definition1: The « Riemann zeta function », is a function of the complex variable $z = Re(z) + iIm(z)$ $(\in \mathbb{C}\backslash\{1\})$ defined as the analytic continuation of the function $\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z}$ (converging for $Re(z) > 1$)

Remark: Riemann showed [27] that the series ζ can be continued analytically to all complex z$\neq$ 1, z=1 being a simple pole such that:$\lim_{z\to1}(z-1)\zeta(z) = 1$.

Definition2 : We call « the Riemann conjecture » or «the Riemann hypothesis » or « the Riemann zeta hypothesis », the following assertion : " the zeros points z (i. e : the points $z \in \mathbb{C}\backslash\{1\}$ such that : $\zeta(z) = 0$) of the Riemann zeta function ζ (see definition1), which are not the trivial zeros $z_k = -2k, k \in \mathbb{N}^*$, satisfies : $\mathrm{Re}(z) = \frac{1}{2}$".

Remark: the confirmation or infirmation of the Riemann hypothesis problem is the 8th one of the famous 23 Hilbert problems announced in 1900 [20], and the first among the "7 millennium problems" announced by « the Clay mathematics institute » in 2000[7], and also the first among the 18 Steve Smale problems announced in 1997[32].

Some history: this conjecture was announced by the German mathematician Georg Friedrich Bernard Riemann (1826-1866) in 1859 in a memory presented to the Berlin Academy. He has written: "…it is very probable that all roots [of$\xi(t) = \pi^{-\frac{1}{4}-\frac{it}{2}}\left(it - \frac{1}{2}\right)\left(\int_0^{+\infty} x^{\frac{it}{2}-\frac{3}{4}}e^{-x}dx\right)\zeta\left(it + \frac{1}{2}\right)$, with: t a complex number and $i^2 = -1$], are real.

Certainly one would wish for a stricter proof here; i have meanwhile temporarily put aside the research for this after some fleeting attempts, as it appears unnecessary for the next objective of my investigation." [23][27].
From this date many mathematicians have devoted a lot of time to prove this conjecture but without success so far.

On the eighth of August 1900, the German mathematician David Hilbert (1862-1943) said in his lecture, delivered before the second International congress of mathematicians at Paris, in the 8th point (about the prime numbers problems) : « …of the problems set us by Riemann's paper: « Üeber die Anzahl der primzahlen unter einer gegebenen grosse », it still remains to prove the correctness of an exceedingly important statement of Riemann, viz., that the zeros points of the function $\zeta(s)$ defined by $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$, all have real part $\frac{1}{2}$, except the well-known negative integral real zeros… » [20].

One day, D.Hilbert said: « If I had to be awaked after having slept thousand years, my first question would be: did one show the Riemann hypothesis? »[29].

In 1914, the British mathematician G.H.Hardy (1877-1947) proved that the set $E = \{z \in \mathbb{C} \text{ such that } \zeta(z) = 0 \text{ } and \text{ } Re(z) = \frac{1}{2}\}$ is infinite, but could not give a proof of the Riemann hypothesis [19].

In 1927, E. Landau (1877-1938) showed that if Riemann hypothesis is admitted we can deduce a great number of consequences [24].

In 1940, the French mathematician A. Weil (1906-1998) showed that the Riemann hypothesis is true for the zeta function associated to fields of algebraic functions (on a finite field of constants), by using geometrical arguments [36].

However, A. Weil wrote a day: « when I was young I hoped to prove the Riemann hypothesis. When I became a little old I hoped to can learn and understand a proof of the Riemann hypothesis. Now, I could be satisfied by knowing that there is a proof » [23].

In 1989, J.B.Conrey showed that more than $\frac{2}{5}$ of the zeros of the Riemann zeta function have real part equal to $\frac{1}{2}$ [10].

In 1996, Alain Connes constructed an operator D on a space of functions of adelic variables (i.e.: laying in the ring of adels, i.e. the Cartesian product of $\mathbb{R}$ and all the p-adic fields for p describing the set of prime numbers), for which the spectrum is:$\sigma(D) = \{r \in \mathbb{R}, L\left(\frac{1}{2} + ir\right) = 0\}$, (where $i^2 = -1$, and L is a kind of general zeta function), this permits to deduce the Riemann hypothesis if we can prove that this spectrum contains all the roots of L[8], [9].

In 2000, the "Clay mathematics institute of Cambridge" considered the Riemann hypothesis among the « millennium problems » and devoted a million of Dollars for any one giving a response to the Riemann hypothesis [7].

The Riemann hypothesis was verified, by computer, for higher numerical values.
Indeed in October 2004, X.Gourdon [18] showed that the first 10^{13} non trivial zeros, of the Riemann zeta function, have real part equal to $\frac{1}{2}$, recording then the higher value reached up to now.. Note that numerical methods of Riemann hypothesis verification are based essentially on the intermediate values theorem (by showing that the continuous function:$s \to \pi^{-\frac{s}{2}}(\int_0^{+\infty} t^{\frac{s}{2}-1} e^{-t} dt)\zeta(s)$, (having the same zeros as ζ in the band $0 < Re(z) < 1$ and being real on the critical line), has at least one zero between two points where it takes opposite signs).

On the first of January 2004, Henri de Berliocchi published at Economica a book entitled « infirmation de l'hypothèse de Riemann ». This is simply impossible because (as it is will be shown in the present work) the Riemann hypothesis is true [2].

on the fourteenth of June 2004, Louis De Branges De Bourcia announced that he had proved the generalized Riemann hypothesis, but according to Eric Weisstein: « De Branges has written a number of papers discussing a potential approach to the generalized Riemann hypothesis… and in fact claiming to prove the generalized Riemann hypothesis..; but no actual proofs seem to be present in this papers. Furthermore, Conrey and Li prove a counter example to the De Branges's approach, which essentially means that the theory developed by De Branges, is not viable» [11] [37].

Recently On the twenty sixth of July 2015 the French mathematician Cédric Villani, director of the "Institut de Henri Poincaré" in Paris, declared to the French Journal "Journal de Dimanche" (JDD) [34] that proving the veracity of the Riemann hypothesis is almost an impossible mission. Our present work shows the contrary.

Recently also on the twenty seventh of October 2015 the Italian mathematician Agostino Prastaro published in arxiv [26] a paper entitled “the Riemann hypothesis proved” in which he announce that “the Riemann hypothesis is proved by quantum-extending the zeta Riemann function to a quantum mapping between quantum 1-spheres with quantum algebra $A = \mathbb{C}$ ...algebraic topologic properties of quantum-complex manifolds and suitable bordism groups of morphisms in the category…of quantum-complex manifolds are utilized” [26]. In any case his approach is different of my approach.

For other current and relevant references to the Riemann hypothesis see: H.Berliocchi [3] (2/27/2014), Peter Borwein [4] (2008), Marcus du Sautoy [12] (8/14/2014), some works of Ivan Fesenko on zeta functions and theta functions [14] (2008), [15] (2010), [16] (2012) and [17] (2015), Barry Mazur/William Stein [25] (2015), D.Rockmore [28] (2006), Ronald Van Der Veen /Jan Van De Craats [33] (1/3/2011), Matthew Watkins [35] (11/17/2015) and its references (Which gives the main papers proving and disproving the Riemann hypothesis, but it seems that no one is convincing)…

The note: the purpose of this original research paper is to prove that the Riemann hypothesis is true by showing that its Balazard equivalent form, which I prove here, is true. I will only use elementary tools of mathematics by exploiting the topological properties of the functions:

$$s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi} \text{ and } s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}.$$

The note is organized as follows. The §1 is an introduction containing the necessary definitions and some history. The §2 contains materials and methods presenting the results needed in the proof of the main theorem and the methods used. The §3 contains the proof of Riemann hypothesis. In §4 we give the conclusions. Finally we give references for further reading.

Motivations: At last, I want to say that, in 1990 when I was preparing my first thesis in the “Pierre et Marie Curie” University in Paris, the lecture of the D. Hilbert’s “mathematical problems” [20] motivated in me the interest of Riemann hypothesis and the obsession of proving it. So this work is the fruit of 26 years of continuous reflection.

INGREDIENTS OF THE PROOF

Materials: We will need the following facts and results in the proof of our main result.

*Notation: for $a, b \in \mathbb{R}$ we denote by $[a, b] = \{t \in \mathbb{R}$ such that: $a \le t \le b\}$
$[a, b[= \{t \in \mathbb{R}$ such that: $a \le t < b\}$ and $]a, b[= \{t \in \mathbb{R}$ such that: $a < t < b\}$

* Recall that an integer p is prime if its divisors are only 1 and p.

*Let: $\mathbb{P} = \{p \in \mathbb{N}, p \text{ prime}\}$, so $\mathbb{P} = \{2,3,5,7,11,13,17,19,23,29, \dots\}$

* Euclid (3rd century before J.C) [13] has showed that $\mathbb{P}$ is a strictly increasing sequence $(p_m)_{m\ge1}$.

*Let for $x \in \mathbb{R}^+$: $\mathbb{P}_x = \{p \in \mathbb{P}, p \le x\}$ and $\pi(x) = card(\mathbb{P}_x)$ the number of its elements. One has: $\mathbb{P}_x = \emptyset$ *and* $\pi(x) = 0$ for $x \in [0,2[$.

*If $[x]$ denotes the integer part of the real x (I.e. the single integer $m = [x]$ such that: $m \le x < m + 1$), one has: $\mathbb{P}_x = \mathbb{P}_{[x]}$ and $\pi(x) = \pi([x])$.

***Proposition1:** (see [38]) If we Let: $\int_0^x \frac{dt}{\ln(t)} = \int_2^x \frac{dt}{\ln(t)} + \int_0^2 \frac{dt}{\ln(t)}$ for $x \ge 2$, then: $\int_0^2 \frac{dt}{\ln(t)} = \lim_{\epsilon\to0}(\int_0^{1-\epsilon} \frac{dt}{\ln(t)} + \int_{1+\epsilon}^2 \frac{dt}{\ln(t)})$ is a positive constant such that:

$$\int_0^2 \frac{dt}{\ln(t)} = 1.0451637801174927848445888891194131365226155781512 \dots$$

***Proposition2:** (See [31],p 60) we have: $\sum_{n=0}^{p} s^n = \frac{1-s^{p+1}}{1-s}$ and $\frac{1}{1-s} = \sum_{n=0}^{+\infty} s^n$ for $|s| < 1$.

***Proposition3:** (Term by term integration, see [21], p 148 and p155) Let $(f_n)_{n\ge0}$ a sequence of measurable functions on $E \subset (X, \Sigma, \mu)$ (a measure space) taking their values in $\mathbb{C}$, such that it exists a summable function g satisfying: $\forall p \in \mathbb{N} \forall x \in E$ $\left|\sum_{n=0}^{p} f_n(x)\right| \le g(x)$, *then:* $\int_E \sum_{n=0}^{+\infty} f_n(x) d\mu(x) = \sum_{n=0}^{+\infty} \int_E f_n(x) d\mu(x)$

***Proposition4:** (Balazard [1] equivalent form of the Riemann hypothesis inspired from the Koch relation [22] as sharpened by the Schoenfeld relation [30]) the two propositions below are equivalents:

(1) The Riemann hypothesis

(2) $\forall x \ge 2657$ $|\pi(x) - \int_0^x \frac{dt}{\ln(t)}| \le \frac{1}{8\pi}\sqrt{s}\ln(s)$

Where: $\pi(x)$ is as defined above, π =the perimeter of the circle of diameter1, $\ln(x)$ is the Napier logarithm and $\int_0^x \frac{dt}{\ln(t)}$ is as defined in proposition1 above.

Proof: (Of proposition4)

$*$ (1) $\Rightarrow$ (2):
This implication is obtained from the Schoenfeld work [30].
*(2) $\Rightarrow$ (1)
This implication follows from proposition4.1, proposition 4.2, proposition4.3, proposition 4.4, Proposition 4.5, proposition 4.6 and proposition 4.7below:

Proposition4.1: We have, for $Re(s) > 1$: $\ln(\zeta(s)) = s\int_2^{+\infty}\frac{\pi(u)du}{u(u^s-1)}$

See: https://fr.wikipedia.org/wiki/Fonction_zeta_de_Riemann [39]

Proposition4.2: We have:

$$s\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^s-1)}du = -\ln(1-\frac{1}{2^s})\int_0^2\frac{dt}{\ln(t)} - \int_2^{+\infty}\frac{\ln(1-u^{-s})du}{\ln(u)} \quad \text{for } Re(s) > 1.$$

Proof: (Of proposition 4.2)

*By proposition 2, for $Re(s) > 1$: $\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}du}{u^{s+1}(1-u^{-s})} = \sum_{n=0}^{+\infty}\int_2^{+\infty}u^{-(n+1)s-1}\int_0^u\frac{dt}{\ln(t)}du$

*So, considering the integration by parts:

$f'(u) = u^{-s(n+1)-1} \quad f(u) = -\frac{u^{-s(n+1)}}{s(n+1)}$

$g(u) = \int_0^u\frac{dt}{\ln(t)} \quad g'(u) = \frac{1}{\ln(u)}$

We have: $\int_2^{+\infty}u^{-s(n+1)-1}\int_0^u\frac{dt}{\ln(t)} = -[\frac{u^{-s(n+1)}\int_0^u\frac{dt}{\ln(t)}}{s(n+1)}]_2^{+\infty} + \frac{1}{s(n+1)}\int_2^{+\infty}\frac{u^{-s(n+1)}du}{\ln(u)}$

$=\frac{2^{-s(n+1)}\int_0^2\frac{dt}{\ln(t)}}{s(n+1)} + \frac{1}{s(n+1)}\int_2^{+\infty}\frac{u^{-s(n+1)}}{\ln(u)}du$

*Then, by proposition 3, we have:

$s\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^s-1)} = \int_0^2\frac{dt}{\ln(t)}\sum_{n=0}^{+\infty}\frac{1}{(n+1)2^{s(n+1)}} + \sum_{n=0}^{+\infty}\frac{\int_2^{+\infty}\frac{u^{-s(n+1)}}{\ln(u)}du}{n+1}$

$=-\ln(1-\frac{1}{2^s})\int_0^2\frac{dt}{\ln(t)} - \int_2^{+\infty}\frac{\ln(1-u^{-s})du}{\ln(u)}$

Proposition4.3: We have:

(1)$\sum_{n=1}^{+\infty}\frac{N(s)^{-2Re(s)(n+1)+1}\ln(N(s))-(2657)^{-2Re(s)(n+1)+1}\ln(2657)}{-2Re(s)(n+1)+1} = -\ln(N(s))\int_{N(s)}^{+\infty}\frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1} + \ln(2657)\int_{2657}^{+\infty}\frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1}$

(2)$\sum_{n=1}^{+\infty}\frac{(M)^{-2Re(s)(n+1)+1}}{(-2Re(s)(n+1)+1)^2} = 2Re(s)\int_M^{+\infty}\frac{\ln(t)dt}{(t^{2Re(s)}-1))^2} + 2Re(s)\int_M^{+\infty}\frac{t^{-2Re(s)}\ln(t)dt}{t^{2Re(s)}-1} - (1+\ln(M))\int_M^{+\infty}\frac{t^{-2Re(s)}dt}{(t^{2Re(s)}-1)} - \frac{M^{-2Re(s)+1}\ln(M)}{M^{2Re(s)}-1}$

Proof: (Of proposition4.3)

(1)*We have: for $M = 2657\ or\ N(s)$

$\sum_{n=1}^{+\infty}\frac{M^{-2Re(s)(n+1)+1}}{-2Re(s)(n+1)+1} = -\int_M^{+\infty}t^{-2Re(s)}dt\sum_{n=1}^{+\infty}t^{-2nRe(s)} = -\int_M^{+\infty}t^{-2Re(s)}\left(\frac{1}{1-t^{-2Re(s)}}-1\right) = -\int_M^{+\infty}\frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1}$

*The result follows.

Remarks :((i) $\frac{1}{t^{2Re(s)}(t^{2Re(s)}-1)}$ is equivalent in the neighbourhood of $+\infty$ to $t^{-4Re(s)}$ so the integral converges for $Re(s) > \frac{1}{4}$

(ii)If we replace s by $1-s$ the integral obtained is convergent for: $0 < Re(s) < \frac{3}{4}$

(2) This second assertion is obtained by derivation of the precedent relatively to $x = Re(s)$.

$\left(\sum_{n=1}^{+\infty}\frac{M^{-2x(n+1)+1}}{-2x(n+1)+1}\right)' = 2\sum_{n=1}^{+\infty}\frac{(n+1)M^{-2x(n+1)+1}}{(-2x(n+1)+1)^2} - 2\ln(M)\sum_{n=1}^{+\infty}\frac{(n+1)M^{-2x(n+1)+1}}{-2x(n+1)+1}$

$=-\frac{1}{x}\sum_{n=1}^{+\infty}\frac{(-2x(n+1)+1-1)M^{-2x(n+1)+1}}{(-2x(n+1)+1)^2} + \frac{\ln(M)}{x}\sum_{n=1}^{+\infty}\frac{(-2x(n+1)+1-1)M^{-2x(n+1)+1}}{-2x(n+1)+1}$

$=-\frac{1}{x}\sum_{n=1}^{+\infty}\frac{M^{-2x(n+1)+1}}{-2x(n+1)+1} + \frac{1}{x}\sum_{n=1}^{+\infty}\frac{M^{-2x(n+1)+1}}{(-2x(n+1)+1)^2} + \frac{\ln(M)}{x}\sum_{n=1}^{+\infty}M^{-2x(n+1)+1} - \frac{\ln(M)}{x}\sum_{n=1}^{+\infty}\frac{M^{-2x(n+1)+1}}{-2x(n+1)+1}$

$=\frac{(1+\ln(M))}{x}\int_M^{+\infty}\frac{t^{-2x}dt}{t^{2x}-1} + \frac{1}{x}\sum_{n=1}^{+\infty}\frac{M^{-2x(n+1)+1}}{(-2x(n+1)+1)^2} + \frac{\ln(M)}{x}\left(\frac{M^{-2x+1}}{M^{2x}-1}\right) = -\left(\int_M^{+\infty}\frac{t^{-2x}dt}{t^{2x}-1}\right)'$

$=2\int_M^{+\infty}\frac{t^{-2x}\ln(t)dt}{(t^{2x}-1)^2} + 2\int_M^{+\infty}\frac{t^{-2x}\ln(t)dt}{t^{2x}-1}$

Proposition4.4: For $Re(s) > 1$, we have:

$$\int_{2657}^{+\infty}\frac{\ln(u)\,du}{\sqrt{u}|u^s-1|} \leq \int_{2657}^{+\infty}\frac{\ln(u)\,du}{\sqrt{u}(u^{Re(s)}-1)}$$

Proof: (Of proposition4.4)
*We have: $|u^s - 1| \geq |u^s| - 1 = u^{Re(s)} - 1 > (2657)^{Re(s)} - 1 > 0$ for $Re(s) > 1$
*The result follows

Proposition4.5: We have: $\exists a(s) > 0, b(s) > 0, A(s)$ some constants such that:
$\int_{2657}^{N(s)} \frac{\ln(u)du}{(u^{2Re(s)}-1)} = \frac{a(s)}{2Re(s)-1} + \frac{b(s)}{(2Re(s)-1)^2} + A(s)$ for $1 > Re(s) > \frac{1}{2}$
Proof: (Of proposition4.5)

*We have:
$\int_{2657}^{N(s)} \frac{\ln(u)du}{(u^{2Re(s)}-1)} = \int_{2657}^{N(s)} \frac{\ln(u)}{u^{2Re(s)}(1-u^{-2Re(s)})} = \sum_{n=0}^{+\infty} \int_{265è}^{N(s)} u^{-2Re(s)(n+1)} \ln(u)\, du$
*Consider the integration by parts:
$f'(u) = u^{-2Re(s)(n+1)}$ $\quad f(u) = \frac{u^{-2Re(s)(n+1)+1}}{-2Re(s)(n+1)+1}$

$g(u) = \ln(u)$ $\quad g'(u) = \frac{1}{u}$
*We have:
$\int_{2657}^{N(s)} u^{-2Re(s)(n+1)} \ln(u) = [\frac{u^{-2Re(s)(n+1)+1}\ln(u)}{-2Re(s)(n+1)+1}]_{2657}^{N(s)} - \frac{1}{-2Re(s)(n+1)+1} \int_{2657}^{N(s)} u^{-Re(s)(n+1)} du$
$= \frac{N(s)^{-2Re(s)(n+1)+1}\ln(N(s)) - \ln(2657)(2657)^{-2Re(s)(n+1)+\frac{1}{2}}}{-2Re(s)(n+1)+1} - \frac{N(s)^{-2Re(s)(n+1)+1} - (2657)^{-2Re(s)(n+1)+1}}{(-2Re(s)(n+1)+1)^2}$
*So by proposition4.3:
$\int_{2657}^{N(s)} \frac{\ln(u)du}{(u^{2Re(s)}-1)} = \sum_{n=0}^{+\infty} \frac{N(s)^{-2Re(s)(n+1)+1}\ln(N(s)) - 2657^{-2Re(s)(n+1)+1}}{-2Re(s)(n+1)+1} - \sum_{n+0}^{+\infty} \frac{N(s)^{-2Re(s)(n+1)+1} - 2657^{-2Re(s)(n+1)+1}}{(-2Re(s)(n+1)+1)^2}$
$= \frac{\ln(N(s))N(s)^{-2Re(s)+1} - \ln(2657)(2657)^{-Re(s)+1}}{-2Re(s)+1} - \frac{N(s)^{-2Re(s)+1} - (2657)^{-2Re(s)+1}}{(-2Re(s)+1)^2} - \ln(N(s)) \int_{N(s)}^{+\infty} \frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1} +$
$\ln(2657) \int_{2657}^{+\infty} \frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1} - (2Re(s) \int_{N(s)}^{+\infty} \frac{t^{-2Re(s)}\ln(t)dt}{t^{2Re(s)}-1} + 2Re(s) \int_{M}^{+\infty} \frac{\ln(t)dt}{(t^{2Re(s)}-1)^2} - (1 +$
$\ln(N(s)) \int_{N(s)}^{+\infty} \frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1} - \frac{N(s)^{-2Re(s)+1}\ln(N(s))}{N(s)^{2Re(s)}-1}) + 2R(s) \int_{2657}^{+\infty} \frac{t^{-2Re(s)}\ln(t)dt}{t^{2Re(s)}-1} + 2Re(s) \int_{2657}^{+\infty} \frac{\ln(t)dt}{(t^{2Re(s)}-1)^2} - (1 +$
$\ln(2657)) \int_{2657}^{+\infty} \frac{t^{-2Re(s)}dt}{t^{2Re(s)}-1} - \frac{2657^{-2Re(s)+1}\ln(2657)}{2657^{2Re(s)}-1}$
$= \frac{a(s)}{2Re(s)-1} + \frac{b(s)}{(-2Re(s)+1)^2} + A(s)$, where $A(s)$ is a constant such that $|A(s)| < +\infty$ for $\frac{1}{2} \leq Re(s) < 1$
Where: $a(s) = -(\ln\left((N(s))N(s)^{-2Re(s)+1} - \ln(2657)\, 2657^{-2Re(s)+1}\right) > 0$
And $b(s) = -\left(N(s)^{-2Re(s)+1} - 2657^{-2R(e(s)+1)}\right) > 0$ because: $N(s) > 2657\ and\ Re(s) > \frac{1}{2}$

Propostion4.6: (1) We have for $Re(s) > 1$ $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s}$ with $\Omega(n)$ the repetitive number of prime factors of n (Recall that : $\lambda(n) = (-1)^{\Omega(n)}$ is called the Liouville number)
(3)$\sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^{\Omega(n)}}{n^s} = (\frac{1}{2^{s-2}} - 1) \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s}$ and $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = (\frac{1}{2^{s-1}} - 1) \sum_{n=1}^{+\infty} \frac{1}{n^s}$
(2) We have
For $1 > Re(s) > \frac{1}{2}$: $\ln(\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s}) = \ln(\zeta(2s)) - \ln(\sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^{\Omega(n)}}{n^s}) + \ln\left(\frac{1}{2^{s-1}} - 1\right) + \ln(\frac{1}{2^{s-2}} - 1)$

Proof: (Of proposition4.6)
(1)See [36]
(2)Because $\Omega(2n) = \Omega(n) + 1$, we have:
$\sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^{\Omega(n)}}{n^s} = \frac{1}{2^s} \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(2n)}}{n^s} - \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(2n+1)}}{(2n-1)^s}$
$= \frac{1}{2^s} \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(2n)}}{n^s} - (\sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s} - \frac{1}{2^s} \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(2n)}}{n^s}$
$= \frac{1}{2^{s-1}} \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(2n)}}{n^s} - \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s} = (\frac{1}{2^{s-2}} - 1) \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s}$
(3)We have from (1) and (2) for $Re(s) > 1$:
$$\frac{\left(\frac{1}{2^{s-1}} - 1\right)\zeta(2s)}{\left(\frac{1}{2^{s-1}} - 1\right)\zeta(s)} = \frac{(\frac{1}{2^{s-2}} - 1) \sum_{n=1}^{+\infty} \frac{(-1)^{\Omega(n)}}{n^s}}{(\frac{1}{2^{s-2}} - 1)} \Leftrightarrow \frac{\left(\frac{1}{2^{s-1}} - 1\right)\zeta(2s)}{\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s}} = \frac{\sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^{\Omega(n)}}{n^s}}{(\frac{1}{2^{s-2}} - 1)}$$
All the series in the precedent relation being convergent for $\frac{1}{2} < Re(s) < 1$, this relation can be extended to this band. So, by taking the logarithm, we have for: $Re(s) \in]\frac{1}{2}, 1[$:

ISSN 2349-0292
Impact Factor 2.675

$$\ln(\zeta(2s)) = \ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s}) + \ln(\sum_{n=1}^{+\infty}\frac{(-1)^n(-1)^{\Omega(n)}}{n^s}) - \ln\left(\frac{1}{2^{s-1}}-1\right) - \ln(\frac{1}{2^{s-2}}-1)$$

Proposition4.7: (1) we have for any complex $s \neq 0$ *and* 1:

$$\zeta(1-s) = 2(2\pi)^{-2s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s)\zeta(s)$$

Where $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt$ is the gamma function (converging for Re(s)>0)

(2)We have For $0 < Re(s) < \frac{1}{2}$

$\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^{1-s}}) = \ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s}) + \ln\left(2(2\pi)^{-2s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s)\right) + \ln\left(\frac{1}{2^{-s}}-1\right) - \ln(\frac{1}{2^{s-1}}-1)$

Proof: (of proposition 4.7)

(1)See [27] and [39].

(2) The result is deduced from (1) by extension.

Return to the proof of proposition4:

*Suppose: $\forall u \geq 2657 \quad |\pi(u) - \int_0^u \frac{dt}{\ln(t)}| \leq \frac{\sqrt{u}\ln(u)}{8\pi}$ and show that:

$$0 < Re(s) < 1 \text{ and } \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s} = 0 \Rightarrow Re(s) = \frac{1}{2}$$

First case: Suppose $\frac{1}{2} < Re(s) < 1$

*For $1 > Re(s) > \frac{1}{2}$, we have: $Re(2s) > 1$, so by the proposition 4.1… Proposition4.6

$$-\frac{1}{2} + |\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s})| - |\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n(-1)^{\Omega(n)}}{n^s}| - \ln\left(\frac{1}{2^{s-1}}-1\right) - \ln(\frac{1}{2^{s-2}}-1)| - |2s\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}|$$

$\leq -\frac{1}{2} + |\ln(\zeta(2s))| - \left|2s\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du\right| < |\ln(\zeta(2s)) - 2s\int_2^{+\infty}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du| = |2s\int_2^{+\infty}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du|$

$=\left|2s\int_2^{2657}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du + 2s\int_{2657}^{+\infty}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du\right| \leq \left|2s\int_2^{2657}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du\right| + 2s\int_{2657}^{+\infty}|\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du|$

$\leq \left|2s\int_2^{2657}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}\right| + \frac{|2s|}{8\pi}\int_{2657}^{+\infty}\frac{\sqrt{u}\ln(u)du}{|u(u^{2s}-1)|} \leq |2s\int_2^{2657}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}| + \frac{|2s|}{8\pi}\int_{2657}^{+\infty}\frac{\ln(u)}{\sqrt{u}(u^{2Re(s)}-1)}$

*Now, by the absurd argument, I deduce that: $\exists N(s)$ an integer > 2657 such that:

$|\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s})| < \frac{1}{2} + \left|2s\int_{2657}^{N(s)}\frac{\int_0^u\frac{dt}{\ln(t)}}{u(u^{2s}-1)}du\right| + \left|s\int_2^{2657}\frac{\pi(u)-\int_0^u\frac{dt}{\ln(t)}}{u(u^s-1)}du\right| + |\ln(\sum_{n=1}^{N(s)}\frac{(-1)^n(-1)^{\Omega(n)}}{n^s}) - \ln\left(\frac{1}{2^{s-1}} - 1\right)\left(\frac{1}{2^{s-2}}-1\right) + \frac{2|s|}{8\pi}\int_{2657}^{N(s)}\frac{\sqrt{u}\ln(u)du}{u(u^{2Re(s)}-1)}| = B(s) + \frac{2|s|}{8\pi}\int_{2567}^{N(s)}\frac{\ln(u)du}{\sqrt{u}(u^{2Re(s)}-1)}$

$\leq B(s) + \frac{|2s|}{8\pi}\int_{2657}^{N(s)}\frac{\ln(u)du}{u^{2Re(s)}-1}$ (Because: $\frac{1}{\sqrt{u}} \leq 1$)

$=A(s) + B(s) + \frac{a(s)}{2R(s)-1} + \frac{b(s)}{(-2Re(s)+1)^2}$ (By proposition 4.5 with: $a(s) > 0, b(s) > 0$)

$(\frac{a(s)}{2Re(s)-1}$ is >0 because $a(s) > 0$ and $\frac{1}{2} < Re(s) < 1$

*So for the first case:

$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s} = 0 \Rightarrow |\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s})| = +\infty \Rightarrow \frac{a(s)}{2Re(s)-1} + \frac{b(s)}{(-2Re(s)+1)^2} = +\infty \Rightarrow 2Re(s) - 1 = 0 \Rightarrow Re(s) = \frac{1}{2}$

Second case: Suppose 0< $Re(s) < \frac{1}{2}$

*We have: $2Re(1-s) > 1$

*So, reasoning as in the first case, it is sufficient to replace, in its last relation:s *by* $1-s$,

*Then:$|\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^{1-s}})| < A(1-s) + B(1-s) + \frac{a(1-s)}{2R(1-s)-1} + \frac{b(1-s)}{(-2Re(1-s)+1)^2}$

Where: $a(1-s) > 0$ and $b(1-s) > 0$

*Finally by the second assertion of proposition4.7, we have:

$|\ln(\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s})| < A(1-s) + B(1-s) + |\ln(2(2\pi)^{-2s}\frac{\frac{1}{2^{-s}}-1}{\frac{1}{2^{s-1}}-1}\Gamma(s)\cos(\frac{\pi}{2}s)| + \frac{a(1-s)}{2Re(1-s)-1} + \frac{b(1-s)}{(2Re(1-s)-1^2}$

*So for the second case also:

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^s} = 0 \Rightarrow Re(s) = \frac{1}{2}$$

Conclusion: We have showed that:

$$\forall u \geq 2657 \quad \left|\pi(u) - \int_0^u \frac{dt}{\ln(t)}\right| \leq \frac{\sqrt{s}\ln(s)}{8\pi} \Rightarrow \text{the Riemann hypothesis is true}$$

***Proposition5:** (The intermediate values theorem) let $f:[a,b] \to \mathbb{R}$ be a function. If $(i) f$ is continuous (ii)$\inf\big(f(a), f(b)\big) \leq c \leq \sup\big(f(a), f(b)\big)$, then: $\exists d \in [a,b]$ such that :

$$f(d) = c$$

The form used in numerical verifications of the Riemann hypothesis is the following: if (i)$a < b$ $(ii) f$ $is\ continuous$ $(iii) f(a)f(b) < 0$, then:$\exists c \in]a,b[$ such that:$f(c) = 0$.

***Definition/Proposition6:** Let (X,T) be a topological space and $f: X \to \mathbb{R}$ be a function.

(1)We say that f is upper semi-continuous on $x \in X$ (usc on x) if :

$\forall \epsilon > 0 \exists V$ a neighbourhood of x Such that $\forall y \in V \quad f(y) < f(x) + \epsilon$

(2)We say that f is usc on X if it is usc in any point $x \in X$.

(3)We say that f is lower semi-continuous (lsc) on X if $(-f)$ is usc on X.

(4)f Lsc on $X \Leftrightarrow \forall t \in \mathbb{R} \quad F = \{x \in X, f(x) \leq t\}$ is a closed subset of X.

(5)f Continuous on $X \Leftrightarrow f$ is lsc and usc on X.

***Proposition7:** (Recollement principle) let (X,T) and(Y,S) be two topological spaces and $f: X \to Y$ be a function. Let $(A_i)_{i \in I}$ be a family of open subsets of X such that:$\forall i \in I$ $f: A_i \to Y$ is continuous. Then the function $f: \bigcup_{i \in I} A_i \to Y$ is also continuous.

***Proposition8:** the function $f: [2, +\infty[\to \mathbb{N}$ defined by $f(x) = \pi(x)$ is usc on $[2, +\infty[$, it is right continuous in p_k $\forall k \geq 1$ and left discontinuous in p_k $\forall k \geq 2$.

Proof :(Of proposition8)

*Using proposition4, we can see that f is continuous on $\bigcup_{k=1}^{+\infty}]p_k, p_{k+1}[$ because: $\forall k \geq 1$ $\forall x \in]p_k, p_{k+1}[$ $f(x) = \pi(p_k) = k$ is constant so continuous.

* f is right continuous in p_k $\forall k \geq 1$

(Because : $\lim_{h \to 0^+} \pi(p_k + h) = \lim_{h \to 0^+} \pi([p_k + h]) = \lim_{h \to 0^+} \pi(p_k + [h]) = \pi(p_k)$)

*f is left discontinuous in p_k $\forall k \geq 2$

(Because : $\lim_{h \to 0^+} \pi(p_k - h) = \lim_{h \to 0^+} \pi([p_k - h]) = \lim_{h \to 0^+} \pi(p_k + [-h]) = \pi(p_k - 1)$

$= \pi(p_{k-1}) = k - 1 \neq k = \pi(p_k)$)

*Show that: f is usc in p_k $\forall k \geq 2$. Let $\epsilon > 0$

**if $\epsilon \in]0,1] \exists V =]p_k - \epsilon, p_k + \epsilon[$ a neighborhood of p_k such that:$\forall x \in V$

***if $x \in]p_k - \epsilon, p_k[$

$0 < \epsilon \leq 1 \Rightarrow p_k - \epsilon \geq p_k - 1 \geq p_{k-1}$ $\forall k \geq 2 \Rightarrow$

$\pi(x) = \pi(p_{k-1}) = k - 1 < \pi(p_k) = k < \pi(p_k) + \epsilon$

***if $x \in [p_k, p_k + \epsilon[$

$0 < \epsilon \leq 1 \Rightarrow p_k + \epsilon \leq p_k + 1 < p_{k+1} \forall k \geq 2 \Rightarrow \pi(x) = \pi(p_k) < \pi(p_k) + \epsilon$

**likely if $\epsilon \in]1, +\infty[\exists V =]p_k - \frac{1}{\epsilon}, p_k + \frac{1}{\epsilon}[$ a neighbourhood of p_k such that

$$\forall x \in V \quad \pi(x) < \pi(p_k) + \epsilon$$

(Note that because:$0 < \frac{1}{\epsilon} < 1$, the precedent reasoning is also valid)

***Proposition9:** Let (X,T) be a topological space. A part $F \subset X$ is closed if: $O = X \backslash F \in T$. Let $Y \subset X$, we define a topology T_Y on Y (inducted by the topology T), defined by:

$$U \in T_Y \Leftrightarrow \exists O \in T \text{ such that } U = Y \cap O$$

Let $Z \subset Y$. We call adherence of Z relatively to T_Y, denoted $adh_Y(Z)$, the intersection of all closed subsets of Y (for T_Y) containing Z. One has the following elementary result : if X is metrical space, then for $Z \subset Y \subset X, x \in Y$, one has an equivalence between the two assertions :

(i)$x \in adh_Y(Z)$

(ii)$\exists (x_m)_{m \geq 1} \subset Z$ such that $x = \lim_{m \to +\infty} x_m$

***Proposition10:** Any non empty above bounded part A of $\mathbb{R}$, has $\sup(A)$, which is by definition, the smallest above bound.

We have: $\sup(A) \in adh_{\mathbb{R}}(A)$

Methods: The paper is founded essentially on proposition4 and the topological properties of the functions:$s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi}$ and $s \to \pi(s) - \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$

RESULTS AND DISCUSSION

Theorem: We have:

$$\forall x \geq 2657 \quad \left|\pi(x) - \int_0^x \frac{dt}{\ln(t)}\right| \leq \frac{\sqrt{x}\ln(x)}{8\pi}$$

ISSN 2349-0292
Impact Factor 2.675

Proof: (Of the theorem)
The proof will be deed in two steps.
In the first step: I will show that $\forall s \geq 2657\ \ \pi(s) \leq \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$
In the second step: I will show that $\forall s \geq 2657\ \ \pi(s) \geq \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi}$
I-First step:
Lemma1: $\forall\ s \geq 2657$ we have $\pi(s) \leq \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$
Proof: (of lemma1)
Let:
$* \ \varphi(s) = \int_0^s \frac{dt}{\ln(t)} + \frac{\sqrt{s}\ln(s)}{8\pi}$,
$* \ \theta(s) = \pi(s) - \varphi(s)$
$* \ A = \{s \in [2657, +\infty[$ such that : $\theta(s) \leq 0\}$
Let for $n \geq 384$: $A_n = \{s \in [2657, p_n]$ such that $[2657, s] \subset A\}$, $((p_n)_{n\geq 1}$ being the strictly increasing sequence of positive prime integers)
Remark: I will deduce lemma1 from seven claims.
1-In claim1: I will show that $\pi(2657) < \varphi(2657)$
2-In claim2: I will show that $adh_{[2657,+\infty[}(A) \subset A \cup (\bigcup_{k=384}^{+\infty}\{p_k\})$
Claim2 will be deduced from two under claims
2-1-In under claim1: I will show that $\varphi: [2, +\infty[\to [\varphi(2), +\infty[$ is a strictly increasing continuous and derivable function.
2-2-In under claim2: I will show that the function $\theta = \pi - \varphi$ is continuous on the intervals $[p_k, p_{k+1}[\ \ \forall k \geq 1$.
3-in claim3: I will show that $a_n = \sup(A_n)$ exists in $adh_{[2657,+\infty[}(A_n)$
4-In claim4: I will show that $[2657, +\infty[\subset A_n \subset A$
5-In claim5: I will show that $\forall n \geq 384\ \ a_n \leq a_{n+1}$
6-In claim6: I will show that $a_n \in A_n\ or\ a_n = p_n$
Claim 6 will be deduced from two under claims
6-1**In under claim3:** I will show $A_n = A_k$ if $k \geq 384$ is such that $a_n = p_k$
6-2In under claim4: I will show $a_n = p_k = p_n$
7-In claim7: I will show $(2657, +\infty[= A$
Claim7 will be deduced from two claims:
7-1 **In under claim5:** I will show $\forall n \geq 385\ \ p_{n-1} \leq a_n$
7-2 In under claim6: I will show $\forall n \geq 386\ \ [p_{n-2}, p_{n-1}] \subset A$
Give now the proofs.
Claim1: One has: $\pi(2657) = 384 < \varphi(2657) = 415{,}7684926 \ldots$
Proof: (Of claim1)
(1)By the list of prime numbers [6], 2657 is the 384th prime number, so:

$$\pi(2657) = 384$$

(2)*According to [5], we have: $\int_2^{2657} \frac{dt}{\ln(t)} \cong 398.5516531 \ldots$
*So, using proposition1 of Preliminaries, we have:

$$\int_0^{2657} \frac{dt}{\ln(t)} = \int_2^{2657} \frac{dt}{\ln(t)} + \int_0^2 \frac{dt}{\ln(t)} = 398.5516531 \ldots + 1.0451637 \ldots \cong 399.5968169$$

*The result follows, because: $\frac{\sqrt{2657}\ln(2657)}{8\pi} = 16.1716757$ (according to [5])
Claim2: We have: $adh_{[2657,+\infty[}(A) \subset A \cup (\bigcup_{k=384}^{+\infty}\{p_k\})$.
Under claim1:
$\varphi: [2, +\infty[\to [\varphi(2), +\infty[$ is a strictly increasing continuous and derivable function.
Proof: (Of the under claim1)
*Using proposition1, the function: $\int_0^x \frac{dt}{\ln(t)} = \int_2^x \frac{dt}{\ln(t)} + \int_0^2 \frac{dt}{\ln(t)} = \int_2^x \frac{dt}{\ln(t)} + 1.045 \ldots$ is evidently continuous and derivable for $x \geq 2$.
*We have: $\varphi'(x) = \frac{1}{\ln(x)} + \frac{1}{8\pi}\left(\frac{1}{\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}}\right) > 0$ for any $x \geq 2$.
*So, the result follows.
Under claim2: $\theta = \pi - \varphi$ is continuous on $[p_k, p_{k+1}[\ \forall k \geq 1$
Proof: (Of the under claim2)
$*x \in [p_k, p_{k+1}[\Rightarrow \pi(x) = \pi(p_k) = k$ is continuous in x because it is constant.
*Using the under claim1, the under claim2 is, then, proved.
Proof: (Of claim2)

*Let: $x \in adh_{[2657,+\infty[}(A) \subset [2657,+\infty[= \bigcup_{k=384}^{+\infty}[p_k, p_{k+1}[$
* According to proposition9 (of preliminaries), we have $x = \lim_{m\to+\infty} x_m$ with: $x_m \in A\ \forall m \geq 1$ and $x \in [p_k, p_{k+1}[$ for a certain integer $k \geq 384$.
So, we have two cases:
First case: $x = p_k$
Second case: $x \in]p_k, p_{k+1}[$
*$]p_k, p_{k+1}[$ being a neighbourhood of x, one has, by definition of $x = \lim_{m\to+\infty} x_m$:

$$\exists N \in \mathbb{N}^* \text{ such that } \forall m \geq N \quad x_m \in]p_k, p_{k+1}[$$

*θ being continuous on $[p_k, p_{k+1}[$ (by the under claim1), one has:

$$x, x_m \in [p_k, p_{k+1}[\ \ \forall m \geq 1 \ and\ x = \lim_{m\to+\infty} x_m \Rightarrow \lim_{m\to+\infty} \theta(x_m) = \theta(x)$$

*So :

$$\forall m \geq N \quad x_m \in A \Rightarrow \forall m \geq 1 \ \ \theta(x_m) \leq 0 \Rightarrow \lim_{m\to+\infty} \theta(x_m) = \theta(x) \leq 0 \Rightarrow x \in A$$

*Finally, by combining the two cases, one has: $x \in A \cup (\bigcup_{k=384}^{+\infty}\{p_k\})$
*The result follows.
Claim3: $a_n = supA_n$ exists in $adh_{[2657,+\infty[}(A_n)$.
Proof: (Of claim3)
A_n is a non empty part (According to claim1, because: $[2657, p_{384}] = \{2657\} \subset A \Rightarrow 2657 \in A_n$), and bounded above by p_n, so $a_n = \sup(A_n)$ exists in $adh_{[2657,+\infty[}(A_n)$ (According to proposition10).
Claim4: We have: $[2657, a_n[\subset A_n \subset A$.
Proof: (Of claim4)
*We have: $A_n \subset A$, by construction of A_n(because :$s \in A_n \Rightarrow [2657, s] \subset A \Rightarrow s \in A$)
*Let: $s \in [2657, a_n[$. By definition of $\sup(A_n)$, we have

$$s < a_n = \sup(A_n) \Rightarrow \exists t \in A_n \text{such that: } s \leq t$$

*Then: $[2657, s] \subset [2657, t] \subset A\ (because\ t \in A_n) \Rightarrow s \in A_n$.
* The result follows.
Claim5: We have: $\forall n \geq 384\ \ a_n \leq a_{n+1}$
Proof: (Of claim5)
*Suppose that: $\exists n \geq 384$ such that: $a_{n+1} < a_n$. In particular: $a_{n+1} \in A_n$(According to claim4)
*$\exists m \geq 1$ such that: $\frac{1}{m} < a_n - a_{n+1} \Leftrightarrow a_{n+1} < a_n - \frac{1}{m} < a_n \leq p_n < p_{n+1}$
*So, by definition of a_{n+1} and a_n : $[2657, a_n - \frac{1}{m}] \subset A$ and $[2657, a_n - \frac{1}{m}] \not\subset A$
*I.e.: $\left[2657, a_n - \frac{1}{m}\right] \cap [2657,+\infty[\backslash A \neq \emptyset \subset A \cap [2657,+\infty[\backslash A = \emptyset$
*This being contradictory, the result is now proved.
Claim6: We have: $a_n \in A_n$ or $a_n = p_n$
Proof: (Of claim6)
*We have, using claim2 and claim4:
$[2657, a_n[\subset A \Rightarrow$
$adh_{[2657,+\infty[}([2657, a_n[) = [2657, a_n] \subset adh_{[2657,+\infty[}(A) \subset A \cup (\bigcup_{k=384}^{+\infty}\{p_k\})$
*So: $a_n \in A \cup (\bigcup_{k=384}^{+\infty}\{p_k\}) \Rightarrow a_n \in A$ or $\exists k \geq 384$ such that: $a_n = p_k$
*Suppose that we are in the second case:" $\exists k \geq 384$ such that:$a_n = p_k \leq p_n$", we have:
Under claim3: We have:$A_n = A_k$
Proof: (Of under claim3)
*We have: $a_n = p_k \Rightarrow a_n = p_k \geq a_k$
*So: $[2657, a_n]\backslash[2657, p_k] = \emptyset \supset [2657, a_k]\backslash[2657, p_k]$
$\Rightarrow [2657, a_k]\backslash[2657, p_k] = \emptyset \Rightarrow [2657, a_k] = [2657, p_k] \Rightarrow a_k = p_k = a_n$
$\Rightarrow A_k = A_n$
Under claim4: $a_n = p_k = p_n$
Proof: (Of under claim4)
*By the precedent under claim3, we have:

$$a_n = p_k \Rightarrow A_n = A_k \Rightarrow adh_{[2657,+\infty[}(A_k) = adh_{[2657,+\infty[}(A_n)$$

*So: $A_n \subset adh_{[2657,+\infty[}(A_n) \subset [2657, p_n]$
*Then: $adh_{[2657,+\infty[}(A_k)\backslash adh_{[2657,+\infty[}(A_n) = \emptyset \supset adh_{[2657,+\infty[}(A_k)\backslash[2657, p_n]$
$\Rightarrow adh_{[2657,+\infty[}(A_k)\backslash[2657, p_n] = \emptyset$
$\Rightarrow adh_{[2657,+\infty[}(A_k) = [2657, p_k] = [2657, p_n] \Rightarrow p_k = p_n$
Conclusion: The claim 6 is now proved.
Claim7: We have: $[2657,+\infty[= A$

Proof: (Of claim7)
The proof of claim 7 will follow the under claim below.
Under claim5: $\forall n \geq 385 \quad p_{n-1} \leq a_n$
Proof: (Of under claim5)
*Do a recurrence on $n \geq 385$.
*The property is true for $n = 385$, because: $p_{385-1} = p_{384} = 2657 \leq a_{385}$
*Suppose: $p_{n-2} \leq a_{n-1}$ and show that: $p_{n-1} \leq a_n$
*If not, we have: $p_{n-2} \leq a_{n-1} \leq a_n < p_{n-1}$
*Using claim6, we have two cases: $a_n = p_n$ or $a_n \in A$
First case: The case: $a_n = p_n$ cannot occur because:$a_n = p_n < p_{n-1} < p_n$ is impossible.
Second case: We are, so, in the case $a_n \in A$
First Under case: if $a_n = a_{n-1}$
*We have:

$$a_n = a_{n-1} \Rightarrow A_n = A_{n-1} \Rightarrow A_n \backslash A_{n-1} = \emptyset \supset [2657, a_n] \backslash [2657, p_{n-1}[= [a_n, p_{n-1}]$$

*$[a_n, p_{n-1}] = \emptyset$ being impossible (Because $a_n < p_{n-1}$), this first under case cannot occur.
Second under case: if $p_{n-2} \leq a_{n-1} < a_n < p_{n-1}$
*We have: $a_{n-1} < a_n < p_{n-1} \Rightarrow \exists m \geq 1$ such that $a_{n-1} < a_n - \frac{1}{m} < a_n < p_{n-1}$
*So, by definition of a_n and a_{n-1}:

$$[2657, a_n - \frac{1}{m}] \cap [2675, +\infty[\backslash A \neq \emptyset \subset A \cap [2657, +\infty[\backslash A = \emptyset$$

*This being contradictory, the second under case cannot also occur and the under claim5 is now proved.
Under claim6 We have: $\forall n \geq 386 \quad [p_{n-2}, p_{n-1}] \subset A$
Proof: (Of under claim6)
*Using claim 6, we have:$\forall n \geq 384$ $[2657, a_n] \subset A$ or $[2657, p_n[\subset A$
*So, by the under claim 5, we have in any case: $\forall n \geq 385 \quad [2657, p_{n-1}] \subset A$
*In particular: $\forall n \geq 386 \quad [p_{n-2}, p_{n-1}] \subset A$
Conclusion: Using the under claim6, we have:

$$\forall n \geq 386 \quad [p_{n-2}, p_{n-1}] \subset A \Rightarrow \cup_{n=386}^{+\infty} [p_{n-2}, p_{n-1}] = [2657, +\infty[\subset A \subset [2657, +\infty[$$

Then the claim7, and so lemma1, are proved.
The proof of the first step is, then, finished.
II-Second step :
Lemma2: $\forall s \geq 2657$ we have: $\pi(s) \geq \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi}$
Proof: (Of lemma2)
Let: $\psi(s) = \int_0^s \frac{dt}{\ln(t)} - \frac{\sqrt{s}\ln(s)}{8\pi}$, $\omega = \psi - \pi$ and $B = \{s \in [2657, +\infty[, \ \omega(s) \leq 0\}$
Remark: I will deduce lemma2 from eight claims.
1-In claim8: I will show $\psi(2657) < \pi(2657)$
2-In claim9: I will show that $\psi: [2, +\infty[\rightarrow [\psi(2), +\infty[$ is a strictly increasing continuous and derivable function.
3-In claim10: I will show that the function $\omega = \psi - \pi$ is lower semi continuous on $[2, +\infty[$
4-In claim11: I will show that the function ω is continuous on $[p_k, p_{k+1}[\ \forall k \geq 384$
5-In claim12: I will show that B is a closed subset of $[2657, +\infty[$.
6-In claim13: I will show that $\forall n \geq 384 \ \ p_n \in B$
Claim13 will be deduced from two under claims.
6-1-In under claim7: I show that the function $\omega = \psi - \pi$ is strictly increasing on $[p_n, p_{n+1}[$
6-2-In under claim8: if $\eta(n), \eta(n+1) \in [p_n, p_{n+1}[$ are such that $\pi(p_n) = \psi(\eta(n))$ and $\pi(p_{n+1}) = \psi(\eta(n+1))$ then $\eta(n) < \eta(n+1)$ is impossible.
7-In claim14: I will show $\forall n \geq 384 \ \ [2657, p_n] \subset B$
Claim14 will be deduced from five under claims
7-1-In under claim9: I will show that:

$$b_n = supB_n = \sup\{s \in [2657, p_n] such\ that [2657, s] \subset B\} \text{ exists in } B_n$$

7-2-In under claim10: I will show $[2657, b_n[\subset B_n \subset B$
7-3-In under claim11: I will show $B_n = [2657, b_n](\subset B)$
7-4-In under claim12: I will show $\forall n \geq 385 \ \ b_{n-1} \leq b_n$
7-5-In under claim13: I will show $\forall n \geq 385 \ \ [2657, p_{n-1}] \subset [2657, b_n]$
8-In claim15: I will show that $B = [2657, +\infty[$
Give vow the proofs.
Claim8: We have: $\psi(2657) = 383.4252 \ldots < \pi(2657) = 384$
Proof :(Of claim8)
See the proof of claim1.

Claim9: One has: $\psi: [2, +\infty[\to [\psi(2), +\infty[$ is a strictly increasing continuous derivable function.
Proof: (Of claim9)
*We have: $\psi'(s) = \frac{1}{\ln(s)} - \frac{1}{8\pi}\left(\frac{1}{\sqrt{s}} + \frac{\ln(s)}{2\sqrt{s}}\right)$.
*Let: $g(s) = \sqrt{1 + 16\pi\sqrt{s}} - \ln(s) - 1$.
*We have: $g'(s) = \frac{16\pi^2 s - 16\pi\sqrt{s} - 1}{s\sqrt{1+16\pi\sqrt{s}}(4\pi\sqrt{s} + \sqrt{1+16\pi\sqrt{s}})}$ is > 0 for s≥ 2 (because the trinomial $16\pi^2 s^2 - 16\pi s - 1$ is > 0 for $s > 0.34$)
*So g is strictly increasing for $s \geq 2$.
*Then: $g(s) \geq g(2) = 6.79 \ldots > 0$ for $s \geq 2$
*Because:$g(s) > 0 \Leftrightarrow \psi'(s) = \frac{1}{\ln(s)} - \frac{1}{8\pi}\left(\frac{1}{\sqrt{s}} + \frac{\ln(s)}{2\sqrt{s}}\right) > 0$, ψ is effectively strictly increasing for $s \geq 2$.
Claim10: The function $\omega(s) = \psi(s) - \pi(s)$ is lower semi continuous on $[2, +\infty[$.
Proof: (Of claim10)
The result is obtained by combination of the assertion (3) of the Definition/Proposition6 and proposition8. (ψ is continuous and $-\pi$ sci)
Claim11: The function ω is continuous on $[p_k, p_{k+1}[\ \ \forall k \geq 384$
Proof: (Of claim11)
*We have: $s \in [p_k, p_{k+1}[\Rightarrow \pi(s) = \pi(p_k) = k \Rightarrow \omega(s) = \psi(s) - \pi(s) = \psi(s) - k$ is continuous on $[p_k, p_{k+1}[$ (Because according to claim8, ψ is continuous).
*So the result follows.
Claim12: One has: B is a closed subset of $[2657, +\infty[$.
Proof: (Of claim12)
The result is obtained by application of the assertion (4) of the Definition/proposition6 and claim10. Indeed: ωsci $\Rightarrow B = \{s \in [2657, +\infty[, \omega(s) \leq 0\}$ is a closed subset of $[2657, +\infty[$.
Claim13: $\forall n \geq 384\ \ \psi(p_n) \leq \pi(p_n)$ i.e. $\forall n \geq 384\ \ p_n \in B$
Proof: (Of claim13)
*Do a recurrence on $n \geq 384$.
*Using claim8, the property is true for $n = 384$, suppose, so, that it is true for n and show that it is true for $n + 1$.
*If not, we have:$\psi(p_n) \leq \pi(p_n) < \pi(p_{n+1}) < \psi(p_{n+1})$.
*Since ψ is continuous, the intermediate values theorem$\Rightarrow \exists \eta(n), \eta(n+1) \in [p_n, p_{n+1}[$ such that: $\pi(p_n) = \psi(\eta(n)) < \pi(p_{n+1}) = \psi(\eta(n+1))$
(so we have:$p_n \leq \eta(n) < \eta(n+1) < p_{n+1}$, because ψ is strictly increasing, according to claim9).
Under claim7: $\omega = \psi - \pi$ is strictly increasing on $[p_n, p_{n+1}[$.
Proof: (Of under claim7)
As ψ is strictly increasing, according to claim 9, then:
$\forall t \in [p_n, p_{n+1}[\ \ \omega(t) = \psi(t) - \pi(t) = \psi(t) - \pi(p_n)$ $\Rightarrow \omega$ is a continuous (and strictly increasing) function on$[p_n, p_{n+1}[$.
Under claim8: $\eta(n) < \eta(n+1)$ is impossible.
Proof: (Of under claim8)
*We have: $\exists N \geq 1$ such that:$\forall m \geq N\ \ \frac{1}{m} < \eta(n+1) - \eta(n)$.
*So, using the under claim7, we have:
$$\omega(\eta(n)) = 0 < \omega\left(\eta(n) + \frac{1}{m}\right) < \omega(\eta(n) + \eta(n+1) - \eta(n)) = \omega(\eta(n+1)) = 0$$
*This is contradictory, hence the under claim8 is proved.
Conclusion: The claim13 is now proved.
Claim14: We have: $\forall n \geq 384\ \ \ [2657, p_n] \subset B$
Proof: (Of claim14)
For $n \geq 384$, let:$B_n = \{s \in [2657, p_n]$ such that $[2657, s] \subset B\}$
Under claim9: $b_n = \sup(B_n)$ exists in B_n.
Proof: (Of under claim9)
B_n is a non empty part of $[2657, p_n]$ (According to claim8), bounded above (by p_n). So, using proposition10, we have: $b_n = \sup(B_n) \in adh_{[2657,+\infty[}(B_n)$ exists.
Under claim10: We have: $[2657, b_n[\subset B_n \subset B$
Proof: (Of under claim10)
*We have:$B_n \subset B$, by construction of B_n.
*Let: $s \in [2657, b_n[$. By definition of $b_n = supB_n$: $\exists x \in B_n$ such that: $s \leq x$ (because: $\forall x \in B_n$ s$\geq x \Rightarrow s \geq b_n = supB_n$)
*So: $[2657, s] \subset [2657, x] \subset B$ (by definition of x) $\Leftrightarrow s \in B_n$.
*The result follows.

Under claim11: We have: $B_n = [2657, b_n] \subset B$
Proof: (Of under claim11)
*We have: $B_n \subset [2657, b_n]$ by definition of b_n.
*B being a closed subset of [2657,+∞[(According to claim12), we have, using the under claim10:

$$[2657, b_n[\subset B \Rightarrow adh_{[2657,+\infty[}([2657, b_n[) = [2657, b_n] \subset adh_{[2657,+\infty[}(B) = B$$

*So :

$$b_n \in B_n \Rightarrow [2657, b_n] = [2657, b_n[\cup \{b_n\} \subset B_n \cup B_n = B_n \subset [2657, b_n]$$

Under claim12: We have: $\forall n \geq 385 \quad b_{n-1} \leq b_n$
Proof: (Of under claim12)
*If not:
$\exists n \geq 385$ such that $b_n < b_{n-1} \leq p_{n-1} < p_n \Rightarrow b_{n-1} \notin B_n$
$\Rightarrow [2657, p_n] \supset [2675, p_{n-1}] \supset [2675, b_{n-1}] \not\subset B$
*This is contradictory with the under claim11. So, the under claim12 is proved.
Under claim13: We have: $\forall n \geq 385 \quad [2657, p_{n-1}] \subset [2657, b_n]$
Proof: (Of under claim13)
*Do a recurrence on $n \geq 385$.
*The property is true for $n = 385$, because:$[2657, p_{385-1}] = \{2657\} \subset [2657, b_{385}]$
*Suppose that: $[2657, p_{n-2}] \subset [2657, b_{n-1}]$ and show that: $[2657, p_{n-1}] \subset [2657, b_n]$
*If not, we have: $p_{n-2} \leq b_{n-1} \leq b_n < p_{n-1}$
*So, we can distinguish two cases:
First case: $b_{n-1} = b_n$
* We have:$b_{n-1} = b_n \Rightarrow B_{n-1} = B_n$
* So:
$B_{n-1} \subset [2657, p_{n-1}[\Rightarrow B_n \backslash B_{n-1} = \emptyset \supset B_n \backslash [2657, p_{n-1}[$
$\Rightarrow B_n = [2657, b_n] = [2657, p_{n-1}[\Rightarrow [2657, p_{n-1}[\backslash [2657, b_n] =]b_n, p_{n-1}[= \emptyset$
*This is contradictory (Because our hypothesis is :$b_n < p_{n-1}$). So, this case cannot occur.
Second case: $b_{n-1} < b_n < p_{n-1}$.
* We have: $b_n < p_{n-1}$ and $[2657, b_n] \subset B \Rightarrow b_n \in B_{n-1}$
*But, then: $b_{n-1} = supB_{n-1} < b_n \in B_{n-1}$ is impossible (by definition of the sup)
*So, this case also cannot occur.
Conclusion: our hypothesis « $p_{n-2} \leq b_{n-1} \leq b_n < p_{n-1}$ » is not true, and then, we have:

$$p_{n-2} \leq b_{n-1} \leq p_{n-1} \leq b_n$$

Finally: $\forall n \geq 385 \quad [2657, p_{n-1}] \subset [2657, b_n] \subset B$, and the claim14 is now proved.
Claim15: We have: $B = [2657, +\infty[$
Proof: (Of claim15)
*Using under claim11 and under claim 13, we have:
$\forall n \geq 386 \;\; [p_{n-2}, p_{n-1}[\subset [2657, p_{n-1}] \subset [2657, b_n] \subset B$
*So: $\bigcup_{n=386}^{+\infty} [p_{n-2}, p_{p-1}[= [2657, +\infty[\subset B \subset [2657, +\infty[$
Conclusion: The lemma2 is now proved. This finishes the proof of the 2[ed] step and the proof of our main theorem.

CONCLUSIONS
I have proved the Riemann hypothesis via elementary tools of mathematics, because the Riemann hypothesis is equivalent to our main theorem (See the poof of proposition 4 in §2, and the references [22] and [30]).

REFERENCES

[1] Balazard, Michel :un siècle et demi de recherches sur l'hypothèse de Riemann. SMF Gazette, 126, Octobre 2010. Pp 1-24. French Available : smf4.emath.fr/Publications/Gazette/2010/126/smf_gazette-126_7_24.pdf (Accessed on 25.03.2017)
[2] Berliocchi H. Infirmation de l'hypothèse de Riemann. Economica.1/1/2004. French.
[3] Berliocchi H. Contradiction de ZFI, infirmation de l'hypothèse de Riemann. Economica. 27/2/2014. French.
[4] Borwein P. The Riemann hypothesis, a resource for the aficionado and virtuoso Alike. Springer. 2008.
[5] Calculatrice de fonctions. A French site for numerical calculations of functions. Available http://wims.unice.fr/ (Accessed on 08.03.2017
[6] Caldwell Ch. The first 1000 primes, the primes pages. 1994-2004. An English site giving the first prime integers. Available http://www.utm.edu/research/primes/lists/small/1000.txt (Accessed on 08.03.2017)
[7] Clay mathematics institute. The Riemann hypothesis. 2000. Available http://www.claymath.org/millinium/Riemann_hypothesis/ (Accessed on 08.03.2017)

[8] Connes A. Formule de trace en géométrie non commutative et l'hypothèse de Riemann. C.R.Acad. Sci Paris série I .Math ; 1996 :323, N° 12, 1231-1236. French.
[9] Connes A. Réalisation spectrale des zéros de zêta. 1998-1999, Cours au collège de France. Available WWW.alainconnes.org. French. (Accessed on 08.03.2017)
[10] **Conrey** JB. More than two fifths of the zeros of Riemann zeta function are on the critical line. J.Reine angew.math; 1989:399, 1-26.
[11] De Branges L. A proof of the Riemann hypothesis. 2004. Available http://www.math.purdue.edu/ftp_pub/Branges. (Accessed on 08.03.2017)
[12] Du Sautoy M. The music of the primes: searching to solve the greatest mystery in mathematics. 14/8/2012. Available www.amazon.com/the-Riemann-Hypothesis-Greatest-Mathematics/dp/0374250073/ (Accessed on 08.03.2017)
[13] Euclide. Les éléments. Traduction de Peyrard Blanchard. Paris. 1966. French.
[14] Fesenko I. Adelic approach to the zeta functions of arithmetic schemes in dimension two. Moscow .Math.J.B; 2008: 237-317. Available www.maths. nottingham.ac.uk/personal/ibf/ada.pdf (Accessed on 08.03.2017)
[15] Fesenko I. Analysis on arithmetic schemes II. Journal of K-theory; 2010: 5(3), 437-557. Available www.maths.nottingham.ac.uk/personal/ibf/a2.pdf (Accessed on 08.03.2017)
[16] Fesenko I, Ricotto G, Suzuki M. Mean-periodicity and zeta functions. Ann. L'inst. Fourier ; 2012 : (62), 1819-1887. Available www.maths.nottingham.ac.uk/personal/ibf/m-p.pdf (Accessed on 08.03.2017)
[17] Fesenko I. Arithmetic deformation theory via arithmetic fundamental groups and non Archimedean theta functions, notes on the work of Shinichi Mochizuki, Europ. J Math; 2015: (1), 405-440. Available www.maths.nottingham.ac.uk/personal/ibf/notesoniut.pdf (Accessed on 08.03.2017)
[18] Gourdon X. The 10^{13} first zeros of the Riemann zeta function and zeros computation at very large height. 10/2004. Available: numbers.computation.free.fr/constants/Miscellaneous/zetazeros1e13-1e24.pdf (Accessed on 08.03.2017)
[19] Hardy GH. Collected papers. London mathematical society. Oxford. Clarendon Press. 1966-1976.
[20] Hilbert D. Mathematical problems. Lecture delivered before the second international congress of mathematicians at Paris in 8/8/1900.The English version is available: http://babbage.clarku.edu/~djoyce/hilbert/problems.html. (Accessed on 08.03.2017)
[21] The original conference « mathematishe problems » is available: http://www.mathematik.uni_biefeld.de/~Kersten/Hilbert/rede.html (Accessed on 08.03.2017)
[22] Jean, Roger : Mesure et integration. Presses de l'université du Québec. Canada. 1982.
[23] Koch VH. Sur la distribution des nombres premiers. Acta. Math, 1901, vol 24, n°1, 159-182
[24] Available **:** link.springer.com/journal/11511/24/1#page-1 (Accessed on 26.03.2016)
[25] Lachaud G. L'hypothèse de Riemann. La recherche ; 1/10/2001 : N°346. Available www.larecherche.fr/special/math346/lac.html. French. (Accessed on 08.03.2017)
[26] Landau E. Collected works. Edited by L. Mirsky and al. Essen Publisher. Thales Verlag. 1985.
[27] Mazur B, Stein W. Prime numbers and Riemann hypothesis. 2015. Available:wstein.org/rh/ch.pdf (Accessed on 08.03.2017)
[28] Prastaro A. The Riemann hypothesis proved. 27 Oct 2015. Available http://arxiv.org/abs/1305.6845 (Accessed on 08.03.2017)
[29] Riemann GFB. Über die Anzahl der Primzahlen unter einer gegebenen grosse. Monatsberichte der Berliner Akademie. November 1859. German. The English Version by D.R.Wilkins (1998) is Available: http://babbage.clarku.edu/~djoyce/hilbert/problems.html (Accessed on 08.03.2017) See : B.Riemann : œuvres mathématiques. G.Gabay 1990. Réimpression de l'édition de 1898.
[30] Rockmore D. Stalking the Riemann hypothesis. The quest to find the hidden law of primes numbers. New York. Vintage. 2006. Available www.amazon.com/exec/obidos/ASIN/037542136X/did=1111091478/sr=2-1/ (Accessed on 08.03.2017)
[31] Sabbagh K. The Riemann hypothesis: The Greatest Unsolved Problem in Mathematics. Farrar, Straus and Giroux, New York, 2003.
[32] Schoenfeld L. Sharper bounds for the chebyshev functions $\theta(x)$ *and* $\psi(x)$ (part II) Math. Comp.; 1976: Vol 30, N°134, 337-360 Available : www.ams.org/Journals/mcom/1976-30-134/50025-5718-1976-0457374-X/50025-5718-1976-0457374-X.pdf (Accessed on 8.03.2017)
[33] Servien, Claude : Analyse 4. Ellipses/édition marketing S.A 1996. Paris, France.
[34] Smale S. Problèmes mathématiques pour le prochain siècle. Conférence prononcée à l'occasion du 60ème anniversaire d'Arnold, à l'institut Fields, Toronto, Juin 1997. French. This text was published in English in the Mathematical intelligencer, Vol 20, N°2, 1998
[35] Van der Veen R, Van de Craats J. Riemann-Hypothese, een miljoenenprobleem. 1/3/2011. Epsilon Uitgaven. Dutch. See also in Dutch the pdf file entitled:"De Riemann-Hypothese: Een

miljoenenprobleem" by Jan van de Craats leve de Wiskunde, UvA, 11 April 2014. Available https://staff.fnwi.uva.nl/j.vandecraats/RiemannHypotheseLdW.pdf (Accessed on 08.03.2017)

[36] Villani C. Cédric Villani décrypte l'hypothèse de Riemann, sommet des maths. Déclaration donnée par Cédric Villani le 26/7/2015 au journal de Dimanche (JDD). Available www.lejdd.fr/Societe/Sciences/cedric-Villani-decrypte-l-hypothese-de-Riemann-sommet-des-maths-743926/. French. (Accessed on 08.03.2017)

[37] Watkins M. proposed (dis)proofs of the Riemann hypothesis. 17/11/2015. Available www.maths.ex.ac.uk/~mwatkins/zeta/RHproofs.htm (Accessed on 08.03.2017)

[38] Weil A. Œuvres mathématiques. Springer. 1979

[39] Weisstein Eric.W. Riemann Hypothesis. 1999-2005. From mathworld. A wolfram web resource. Available http;//mathworld.wolfram.com/RiemannHypothesis.html (Accessed on 08.**03.2017)**

[40] Weisstein Eric.W. Logarithmic Integral. (1999-2005). From mathworld. A wolfram web resource. Available http://mathworld.wolfram.com/LogarithmicIntegral.html (Accessed on 08.03.2017)

[41] Wikipedia: Fonction zêta de Riemann. Available https://fr.wikipedia.org/wiki/Fonction_zeta_de_Riemann/ . French. (Accessed on 08.03.2017)