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SANDWICH THEOREMS FOR GENERALIZED INTEGRAL OPERATOR $L^{\delta}_{q,s}(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s)$

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ABSTRACT

We introduce some applications of first order differential subordination and superordination to obtain sufficient conditions for generalized integral operator to satisfy

$$q_1(z) \prec \frac{z[L^{\delta}_{q,s}(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s)f(z)]'}{\Phi[L^{\delta}_{q,s}(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s)f(z)]} \prec q_2(z)$$

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INTRODUCTION

Let $\mathcal{H}$ be the class of functions analytic in U and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let A be the subclass of $\mathcal{H}$consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$. Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0) = 0$ and $\Phi'(0)>0$. The function $f \in A$ is called $\Phi-$like if

$$\Re\left\{\frac{zf'(z)}{\Phi(f(z))}\right\} > 0, \qquad z \in U.$$

This concept was introduced by Brickman [2] and established that a function f ∈ A is univalent if and only if f is Φ−like for some Φ.

Definition 1.1. Let Φ be analytic function in a domain containing f(U), Φ(0) =0, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for ω ∈ f(U)−0. Let q(z) be a fixed analytic function in U, q(0) = 1. The function f ∈ A is called Φ−like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z), \qquad z \in U.$$

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G, written F ≺ G, if G is univalent, F(0) = G(0) and F(U) ⊂ G(U). In a more general case, given two functions F(z) and G(z), which are analytic in U, the function F(z) is said to be subordination to G(z) in U if there exists a function h(z), analytic in U with h(0) = 0 and |h(z)| < 1 for all z ∈ U such that F(z) = G(h(z)) for all z ∈ U.

Let $\phi: \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, p ≺ q. If p and $\phi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z), zp'(z))$ then p is called a solution of the differential superordination[6]. An analytic function q is called subordinant of the solution of the differential superordination if q ≺ p.

For $\alpha_i \in \mathbb{C}$ $(j = 1,2,3, \dots, q)$ and $\beta_j \in \mathbb{C} - \{0, -1, -2, \dots\}$ $(j = 1,2,3 \dots, s)$, $\delta < 1$, the generalized integral Operator $L^{\delta}_{q,s}(\alpha_1, \alpha_2 \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s): A \to A$ is defind as

$$L^{\alpha}_{q,s}(\alpha_1, \alpha_2 \dots, \alpha_q; \beta_1, \beta_2, \dots, \beta_s)f(z) = z + \sum_{n=0}^{\infty} \frac{(\beta_1)_{n-1} \dots \dots (\beta_s)_{n-1}}{(\alpha_1)_{n-1} \dots\dots (\alpha_q)_{n-1}} (2 - 2\delta)_{n-1} a_n z^n$$
$$(q \leq s + 1; q, s \in N_0) \qquad (1.1)$$

Where $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a + 1) \dots (a + n - 1)$ for $n \in N = \{1,2, \dots\}$ and 1 when n = 0

This operator is studied by R.S.Khatu and U.H.Naik [3].

For q = s+1 and $\alpha_2 = \beta_1, \ldots, \alpha_q = \beta_s$, we note that $L^0_{q,s}(1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \beta \ldots \beta_s)f(z) = zf'(z)$ and $L^0_{q,s}(2, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \beta \ldots \beta_s)f(z) = f(z)$.

It is well known that,

$\alpha_1 L^\delta_{q,s}(\alpha_1, \alpha_2 \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z) = z[L^\delta_{q,s}(\alpha_1 + 1, \alpha_2 \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z)]' + (\alpha_1 - 1)L^\delta_{q,s}(\alpha_1 + 1, \alpha_2 \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z)$. (1.2)

To make the notation simple, we write,

$$L^\delta_{q,s}[\alpha_1]f(z) = L^\delta_{q,s}(\alpha_1, \alpha_2 \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s)f(z)$$

Also we note that, a special case of $L^0_{q,s}$ is the Noor integral operator[1].

Definition 1.2. Let f ∈ A. Then f ∈ S^*_δ (the starlike subclass of A) if and only if for z ∈ U

$$\Re\left\{\frac{z[L^\delta_{q,s}[\alpha_1]f(z)]'}{L^\delta_{q,s}[\alpha_1]f(z)}\right\} > 0, \quad n \in N_0.$$

In order to prove our subordination and superordination results, we need to the following lemmas in the sequel.

Definition 1.3. [5] Denote by Q the set of all functions f(z) that are analytic and injective on $\overline{U} - E(f)$ where E(f) $:= \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 1.1. [6] Let q(z) be univalent in the unit disk U and θ and φ be analytic in a domain D containing q(U) with φ(w) ≠ 0 when w ∈ q(U). Set Q(z) := zq'(z)φ(q(z)), h(z) := θ(q(z)) + Q(z). Suppose that

1. Q(z) is starlike univalent in U, and
2. $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for z ∈ U.

If θ(p(z)) + zp'(z)φ(p(z)) ≺ θ(q(z)) + zq'(z)φ(q(z)) Then p(z) ≺ q(z) and q(z) is the best dominant.

Lemma 1.2. [7] Let q(z) be convex univalent in the unit disk U and ϑ and ϕ be analytic in a domain D containing q(U). Suppose that

[1] zq'(z)ϕ(q(z)) is starlike univalent in U, and

2. $\Re\left\{\frac{\vartheta'(q(z))}{Q(q(z))}\right\} > 0$ for z ∈ U.

If p(z) ∈ H[q(0), 1]∩Q, with p(U) ⊆ D and ϑ(p(z))+zp'(z)ϕ(z) is univalent in U and ϑ(q(z)) + zq'(z)ϕ(q(z)) ≺ ϑ(p(z)) + zp'(z)ϕ(p(z)) then q(z) ≺ p(z) and q(z) is the best subordinant.

SANDWICH THEOREMS

In this section, and by using Lemmas 1.1 and 1.2, we prove the following subordination and superordination results on the lines of Ibrahim and Darus[4].

Theorem 2.1. Let q(z) ≠ 0be univalent in U such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and

$\Re\left\{1 + \frac{\alpha}{\gamma}q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0, \quad ,\alpha, \gamma \in \mathbb{C}$ and $\gamma \neq 0$ (2.4)

If f ∈ A satisfies the subordination

$\beta\left\{\frac{z[L^\delta_{q,s}[\alpha_1]f(z)]'}{\Phi[L^\delta_{q,s}[\alpha_1]f(z)]}\right\} + \gamma\left\{1 + \frac{z[L^\delta_{q,s}[\alpha_1]f(z)]''}{[L^\delta_{q,s}[\alpha_1]f(z)]'} - \frac{z\Phi'[L^\delta_{q,s}[\alpha_1]f(z)]}{\Phi[L^\delta_{q,s}[\alpha_1]f(z)]}\right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}$,

then $\frac{z[L^\delta_{q,s}[\alpha_1]f(z)]'}{\Phi[L^\delta_{q,s}[\alpha_1]f(z)]} \prec q(z)$ (2.5)

and q(z) is the best dominant.

Proof. Our aim is to apply Lemma 1.1. Setting $p(z) = \frac{z[L^\delta_{q,s}[\alpha_1]f(z)]'}{\Phi[L^\delta_{q,s}[\alpha_1]f(z)]}$.

By computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[L^\delta_{q,s}[\alpha_1]f(z)]''}{[L^\delta_{q,s}[\alpha_1]f(z)]'} - \frac{z\Phi'[L^\delta_{q,s}[\alpha_1]f(z)]}{\Phi[L^\delta_{q,s}[\alpha_1]f(z)]}$$

which yields the following subordination

$$\alpha p(z) + \frac{\gamma z p'(z)}{p(z)} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}, \quad \alpha, \gamma \in \mathbb{C}$$

By setting $\theta(\omega) := \alpha\omega$ and $\phi(\omega) := \frac{\gamma}{\omega}$, $\gamma \neq 0$, it can be easily observed that θ(ω) is analytic in ℂ and ϕ(ω) is analytic in ℂ\{0} and that ϕ(ω) ≠ 0 when ω ∈ ℂ\{0} . Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\gamma z q'(z)}{q(z)}$$

And $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}$, we find that Q(z) is starlike univalent in U and that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \left\{1 + \frac{\alpha}{\gamma}q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0$$

Then the relation (5) follows by an application of Lemma1.1.
When $\Phi(\omega) = \omega$ in Theorem 2.1, we get the following results

Corollary 2.1. Let $q(z) \neq 0$be univalent in U. If q satisfies (2.4) and

$$\alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}$$

then $\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \prec q(z)$ and q(z) is the best dominant.

Corollary 2.2. If $f \in A$ and assume that (2.4) holds then

$$1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} \prec \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Az)(1 + Bz)}$$

implies

$$\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)]} \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \leq B < A \leq 1$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By setting $\alpha = \gamma = 1$ and $q(z) := \frac{1+Az}{1+Bz}$ where $-1 \leq B < A \leq 1$.

Corollary 2.3. If $f \in A$ and assume that (2.4) holds then

$$1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} \prec \frac{1 + z}{1 - z} + \frac{2z}{1 - z^2}$$

implies

$$\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \prec \frac{1 + z}{1 - z}$$

And $\frac{1+z}{1-z}$ is the best dominant.

Proof. By setting $\alpha = \gamma = 1$and $q(z) := \frac{1+z}{1-z}$.

Corollary 2.4. If $f \in A$ and assume that (2.4) holds then

$$1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} \prec e^{Az} + Az$$

Implies $\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \prec e^{Az}$, and e^{Az} is the best dominant.

Proof. By setting $\alpha = \gamma = 1$ and $(z) := e^{Az}$, $|A| < \pi$..

Theorem 2.2. Let $q(z) \neq 0$ be convex univalent in the unit disk U. Suppose that

$$\Re\left\{\frac{\alpha}{\gamma}q(z)\right\} > 0, \quad \alpha, \gamma \in \mathbb{C} \text{ for } z \in U \tag{2.6}$$

and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]} \in \mathcal{H}[q(0), 1] \cap Q$ where $f \in A$,

$$\alpha\left\{\frac{z\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]'}{\Phi\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z\Phi'[L_{q,s}^{\delta}[\alpha_1]f(z)]}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\}$$

is univalent is U and the subordination

$$q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \alpha\left\{\frac{z\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]'}{\Phi\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]}\right\}$$
$$+\gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z\Phi'[L_{q,s}^{\delta}[\alpha_1]f(z)]}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\}$$

holds, then $$q(z) \prec \frac{z\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]'}{\Phi\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]} \tag{2.7}$$

and q is the best subordinant.

Proof. Our aim is to apply Lemma 1.2. Setting

$$p(z) := \frac{z\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]'}{\Phi\left[L_{q,s}^{\delta}[\alpha_1]f(z)\right]}$$

By computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z\Phi'[L_{q,s}^{\delta}[\alpha_1]f(z)]}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}$$

which yields the following subordination

$$q(z) + \frac{\gamma zq'(z)}{q(z)} \prec \alpha p(z) + \frac{\gamma zp'(z)}{p(z)}, \quad \alpha, \gamma \in \mathbb{C}.$$

By setting
$\vartheta(w) := \alpha w$ and $\varphi(w) := \frac{\gamma}{w}$, $\gamma \neq 0$,
it can be easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$ and $\varphi(w) := \frac{\gamma}{w}$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\varphi(w) \neq 0$ when $\omega \in \mathbb{C}\backslash\{0\}$. Also, we obtain

$$\Re\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} = \Re\left\{\frac{\alpha}{\gamma} q(z)\right\} > 0.$$

Then (7) follows by an application of Lemma 1.2.
When $\Phi(\omega) = \omega$ in Theorem 2.2, we obtain the following result
Corollary 2.5. Let $q(z) \neq 0$ be convex univalent in U. If $f \in A$ and

$$\alpha q(z) + \frac{\gamma zq'(z)}{q(z)} \prec \alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}\right\}$$

Then

$$q(z) \prec \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}$$

and q(z) is the best subdominant.
Combining Theorems 2.1 and 2.2 in order to get the following Sandwich result

Theorem 2.3. Let $q_1(z) \neq 0$, $q_2(z) \neq 0$ be convex univalent in the unit disk U satisfy (6) and (4) respectively. Suppose that and $\frac{zq_i'(z)}{q_i(z)}, i = 1,2$ is starlike univalent in U. If $\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \in \mathcal{H}[q_1(0), 1] \cap Q$ where $f \in A$,

$$\alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z\Phi'[L_{q,s}^{\delta}[\alpha_1]f(z)]}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\}$$

is univalent is U and the subordination

$$q_1(z) + \frac{\gamma zq_1'(z)}{q_1(z)} \prec \alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z\Phi'[L_{q,s}^{\delta}[\alpha_1]f(z)]}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\} \prec \alpha q_2(z) + \frac{\gamma zq_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z) \prec \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{\Phi[L_{q,s}^{\delta}[\alpha_1]f(z)]} \prec q_2(z) \tag{2.8}$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.
Combining Corollaries 2.1 and 2.5 in order to get the following Sandwich result
Corollary 2.6. Let $q_1(z) \neq 0, q_2(z) \neq 0$ be convex univalent in the unit disk U satisfy (6) and (4) respectively. Suppose that and $\frac{zq_i'(z)}{q_i(z)}, i = 1,2$ is starlike univalent in U. If $\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ where $f \in A$,

$$\alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\}$$

is univalent is U and the subordination

$$q_1(z) + \frac{\gamma zq_1'(z)}{q_1(z)} \prec \alpha\left\{\frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)}\right\} + \gamma\left\{1 + \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]''}{[L_{q,s}^{\delta}[\alpha_1]f(z)]'} - \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{[L_{q,s}^{\delta}[\alpha_1]f(z)]}\right\} \prec \alpha q_2(z) + \frac{\gamma zq_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z) \prec \frac{z[L_{q,s}^{\delta}[\alpha_1]f(z)]'}{L_{q,s}^{\delta}[\alpha_1]f(z)} \prec q_2(z) \tag{2.9}$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Corollary 2.7. Let the assumption of Theorem 2.3 holds with $q_1(z) = q_2(z) = 1$. Then

$$q_1(z) \prec \frac{z[f(z)]'}{f(z)} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.
Proof. By setting $\Phi(\omega) = \omega, \alpha = \gamma = 1$ and $\delta = 0, \alpha_1 = 2$.
Corollary 2.8. Let the assumption of Theorem 2.3 holds. Then

$$q_1(z) \prec 1 + \frac{z[f(z)]''}{[f(z)]'} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.
Proof. By setting $\Phi(\omega) = \omega, \alpha = \gamma = 1$ and $\delta = 0, \alpha_1 = 1$.

Corollary 2.9. Let the assumption of Theorem 2.3 holds with $q_1(z) \neq 0$, and $q_2(z) \neq 0$. Then

$$q_1(z) \prec \frac{z[f(z)]'}{\Phi[f(z)]} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.
Proof. By setting $\alpha = \gamma = 1$ and $\delta = 0, \alpha_1 = 2$.

Corollary 2.10. Let the assumption of Theorem 2.3 holds with $q_1(z) = q_2(z) = 1$. Then

$$q_1(z) \prec \frac{z[f(z)]'}{\Phi[f(z)]} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.
Proof. By setting $\alpha = \gamma = 1$ and $\delta = 0, \alpha_1 = 2$.

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