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CONFIRMATION OF THE LEGENDRE AND THE EULER CONJECTURES

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ABSTRACT

I show the veracity of the Legendre conjecture, which says that $\forall n \in \mathbb{N}^* \exists p$ a prime integer such that : $n^2 \leq p \leq (n+1)^2$, remained open since 1833 (date of the Legendre extinction). I show also the veracity of the Euler conjecture, which says that there is an infinite number of prime numbers of the form n^2+1, remained open since 1760. I do this by using the Schoenfeld inequality that I have showed in my work [4], and by using the same approach of my papers [5], [6].

KEYWORDS: Legendre conjecture, prime number, semi-prime number, Schoenfeld inequality, prime-counting function, Euler conjecture.
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INTRODUCTION

Definition1 : We call the Legendre conjecture the following assertion « For any not null positive integer $n, \exists p$ a prime integer such that $n^2 \leq p \leq (n+1)^2$ ».

Definition2 : I call the Euler conjecture the assertion « There is an infinite number of prime numbers of the form n^2+1 »

History : The Legendre conjecture was announced, by the French Mathematician Adrien-Marie Legendre (18/9/1752-10/1/1833) (See [11], [12], [13], [14])

The Euler conjecture was announced, in 1760, by the Swiss Mathematician Leonhard Euler (1707-1783) (See [3]).

In 1912, the German Mathematician Edmund Georg Hermann Landau (14/2/1877-19/2/1938) said in his lecture [10] delivered before the fifth international congress of Mathematicians at Cambridge that the following four problems are « unattackable in the actual state of knowledge » :

1-**The Goldbach conjecture** : that is "$\forall n \in \mathbb{N}^* - \{1\} \exists (p,q)$ two prime integers such that :$2n = p + q$".(I have showed, on May 2018, this conjecture in [5])

2-**The Twin Primes conjecture** : that is « it exists an infinite number of prime numbers p such that $p+2$ is prime » (Ihave showed, on July 2018, this conjecture in [6]).

3-**The Legendre conjecture :** that is « $\forall n \in \mathbb{N}^* \exists p$ a prime integer such that :$n^2 \leq p \leq (n+1)^2$ » (It is the subject of the present paper.)

4- **The Euler conjecture :** That is « there is an infinite number of prime numbers of the form n^2+1" (It is also the subject of the present work).

In 1975, the Chinese Mathematician Chen Jingrun (1933-1996) showed in [9] a weak version of the Legendre conjecture, that is « between n^2 and $(n+1)^2$ there is always an integer p which is prime or semi-prime » (Recall that a positive integer p is prime if $p \neq 1$ and if its set of divisors is $\{1,p\}$ and is semi-prime if $\exists n, m$ two prime integers (we can have : $n = m$) such that $p = nm$).

In 1984, the American Polish Mathematician Henryk Iwaniec (Born in 0ctober 9, 1947-) and the Hungarian Mathematician János Pintz (Born in December 20, 1950-) showed, in [8], that there is always a prime integer between : $n - n^{\frac{23}{40}}$ and n.

In 2001, Baker, R.C. ; Hermann, G. ; Pintz, G. and Pintz, J. proved, in [1], that there is always a prime number in the interval $[n, n + O(n^{\frac{21}{40}})]$.

In 2006, The Iranian Mathematician Hassani Mehdi (Born in July 23, 1979-) showed, in [7], that : for infinitely many $n \in \mathbb{N}$, we have :

$$[\frac{1}{2}(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln(n)}) - \frac{(\ln(n))^2}{\ln(\ln(n))}] \leq \pi((n+1))^2 - \pi(n^2)$$

(Where [x] denotes the integer part of the real x).
My present paper gives responses to some questions of Hassani Mehdi in [7].

In 2014, Oliveira e Silva, Tomas Herzog, Sieghied Pardi and Silvio showed, in [15], that the Legendre conjecture is true for any $1 \leq n \leq 4.10^{18}$.

The note : the purpose of the present paper is to show the third and the fourth Landau problems by using the Schoenfeld inequality proved in my work [4] and by using the same methods of my two works [5], [6].

The paper is organized as follows. §1 is an introduction giving the necessary definitions and some History. §2 contains the ingredients of the proofs of our main results. §3 gives the proof of the Legendre conjecture. §4 contains the proof of the Euler conjecture. §6 gives the conclusions and §7 is devoted to the references of the paper for further reading.
Our main theorems are :

Theorem1 : The Legendre conjecture is true.

Theorem2 : The Euler conjecture is true.

INGREDIENTS OF THE PROOFS

For a real number x, I denote by $[x]$ the integer part of x, it means the single integer $m = [x]$ such that : $m \leq x < m + 1$
For A, B two subsets of a set E we note by $A - B = \{a \in E \text{ such that } a \in A \text{ and } a \notin B\}$
If A is a finite subset, we note by $card(A)$ the number of elements of A.

Proposition1 : ([5], [6]) if A, B are finite subsets, then :
(1)$B \subset A \Rightarrow card(A - B) = card(A) - card(B)$
(2)$B \subset A \Rightarrow card(B) \leq card(A)$
(3) if $(A_i)_{1 \leq i \leq m}$ is a finite sequence of finite subsets such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then :

$$card(\bigcup_{i=1}^{m} A_i) = \sum_{i=1}^{m} card(A_i)$$

Recall that a prime integer is a number $p \in \mathbb{N}^* - \{1\}$ having for set of divisors the set $\{1, p\}$.
We note : $\mathbb{P} = \{p \in \mathbb{N}, \text{such that } p \text{ is prime}\} = \{2,3,5,7,11,13,17,19 \dots\}$

Proposition2 :(Euclid (3thd Century before Jésus Christ) [2])$\mathbb{P}$ is an infinite strictly increasing sequence $(p_k)_{k \in \mathbb{N}^*}$.
For $n \in \mathbb{N}$, we note $\mathbb{P}_n = \{p \in \mathbb{P}, \text{such that } p \leq n\}$ and $\pi(n) = card(\mathbb{P}_n)$. The function $n \to \pi(n)$ is called the prime-counting function.

Definition3 : ([5], [6]) if f, g are two real functions defined in the neghborhood of $+\infty$, then :
$f = O(g)$ in the neighborhood of $+\infty \Leftrightarrow \exists A > 0 \exists B \in \mathbb{R}$ such that :

$$\forall t \geq B: |f(t)| \leq A|g(t)|$$

Proposition3 : The prime-counting function $n \to \pi(n)$ is an increasing function but not strictly.

Proposition4 : (The Schoenfeld inequality [4])$\forall n \geq 2657$ $|\pi(n) - \int_0^n \frac{dt}{\ln(t)}| \leq \frac{\sqrt{n}\ln(n)}{8\pi}$
I.e. : $\pi(n) = \int_0^n \frac{dt}{\ln(t)} + O(\sqrt{n}\ln(n))$ for n≥ 2657

Proposition5 : ([5], [6]) (The intermediate value theorem) If $f: [a,b](a < b) \to \mathbb{R}$ is a continuous function. Then : $f(a)f(b) < 0 \Rightarrow \exists c \in]a,b[$ such that : $f(c) = 0$

Proposition6 : we have :
(1) $\lim_{n\to+\infty} x_n = +\infty \Leftrightarrow \forall \epsilon > 0 \exists p \in \mathbb{N} \forall n \geq p: x_n \geq \epsilon$
(2) $\lim_{n\to+\infty} x_n = s \in \mathbb{R} \Leftrightarrow \forall \epsilon > 0 \exists p \in \mathbb{N} \forall n \geq p: |x_n - s| \leq \epsilon$
(3) $\lim_{x\to a} f(x) = f(a) \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0: |x - a| \leq \epsilon \Rightarrow |f(x) - f(a)|$

Proposition7 : if I is a real interval, $a \in I$ and $f: I \to \mathbb{R}$ is a function, we have :
(1)f continuous in $a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$
(2) f continuous in $a \Leftrightarrow (\forall (x_n)$ a real sequence: $\left(\lim_{n\to+\infty} x_n = a\right) \Rightarrow \left(\lim_{n\to+\infty} f(x_n) = f(a)\right))$
(3) f continuous on $I \Leftrightarrow f$ continuous in any element $a \in I$

Proposition8 : ([5],[6]) for $(x_n)_n$ any sequence in $\mathbb{R}$, we have :
(1)$liminf(x_n)$=$sup_{n\in\mathbb{N}} inf_{k\geq n}(x_k)$ exists always in $[-\infty, +\infty]$
(2) If $(y_n)_n$ is a convergent sequence, then :

$$liminf(x_n + y_n) = liminf(x_n) + \lim_{n\to+\infty} y_n$$

(3) If $(x_n) \subset [a,b]$ is bounded, we have : $a \leq liminf(x_n) \leq b$
(4)$liminf(x_n) = s \in \mathbb{R} \Leftrightarrow \begin{cases} \forall \epsilon > 0 \exists n \in \mathbb{N} \forall k \geq n: x_k > s - \epsilon \\ \forall \epsilon > 0 \forall n \in \mathbb{N} \exists k \geq n: x_k < s + \epsilon \end{cases}$
(5) if $y_n \geq 0 \forall n$, we have : $liminf(x_n y_n) \geq liminf(x_n) liminf(y_n)$
(6)$liminf(x_n) = +\infty \Leftrightarrow \lim_{n\to+\infty} x_n = +\infty$

Proposition9 : ([5], [6]) (i) any non empty part $E$of $\mathbb{N}$ has a minimal element : $\min(E)$ characterized by : $\min(E) \in E, \forall x \in E: x \geq \min(E)$ and $\min(E) = 0$ or $\min(E) - 1 \notin E$

(ii) any non empty part $E$of $\mathbb{N}$, bounded above, has a maximal element :$\max(E)$, $\max(E)$ is characterized by : $\max(E) \in E, \forall x \in E: x \leq \max(E)$ and $\max(E) + 1 \notin E$

Proposition10 : ([5], [6]) any sequence $(t_k) \subset [a,b]$ (a bounded real interval), has a convergent subsequence, denoted also(t_k), such that : $\lim_{k\to+\infty} t_k = t \in [a,b]$

Proposition11 : ([5], [6]) (the Hôpital rule), for derivable functions f, g:if $\lim_{t\to+\infty} \frac{f(t)}{g(t)}$ is the indefinite form$\frac{\infty}{\infty}$, then : $\lim_{t\to+\infty} \frac{f(t)}{g(t)} = \lim_{t\to+\infty} \frac{f'(t)}{g'(t)}$, the process can be repeated in the same conditions up determination.

PROOF OF THE LEGENDRE CONJECTURE

Theorem1 : (The Legendre is true)$\forall n \in \mathbb{N}^* \exists p \in \mathbb{P}$ such that :$n^2 \leq p \leq (n+1)^2$

Proof : (of the theorem)
For :$n \in \mathbb{N}^*$, Let :$A_n = \{p \in \mathbb{P}, n^2 \leq p \leq (n+1)^2\}$
It is evident that theorem1 is showed if we prove : $\forall n \in \mathbb{N}^*$ $A_n \neq \emptyset$ $i.e: card(A_n) > 0$
The proof of theorem1 will be deduced from the lemmas below.

Lemma1 : We have :
$card(A_n) = \pi((n+1)^2) - \pi(n^2)$

Proof : (of lemma1)
The result follows by definition of the prime-counting function $n \to \pi(n)$ using the assertion (1) of proposition (1) and the fact that : $A_n = \mathbb{P}_{(n+1)^2} - \mathbb{P}_{n^2}$ (because a prime number cannot be a square)

Lemma2 : for $n \geq \sqrt{2657}$, we have :
$$\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} - \frac{(n+1)\ln(n+1)}{2\pi} \leq card(A_n) \leq \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} + \frac{(n+1)\ln(n+1)}{2\pi}$$
I.e. with the notation of definition3 :
$card(A_n) = \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} + O((n+1)\ln(n+1))$ for $n \geq \sqrt{2657}$
With :$\left|\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right| \leq \frac{1}{4\pi}$

Proof : (of lemma3)
*By the Schoenfeld inequality, we have : for $n \geq \sqrt{2657}$
$$\begin{cases} \int_0^{(n+1)^2} \frac{dt}{\ln(t)} - \frac{\sqrt{(n+1)^2}\ln((n+1)^2)}{8\pi} \leq \pi((n+1)^2) \leq \int_0^{(n+1)^2} \frac{dt}{\ln(t)} + \frac{\sqrt{(n+1)^2}\ln((n+1)^2)}{8\pi} \\ -\int_0^{n^2} \frac{dt}{\ln(t)} - \frac{\sqrt{n^2}\ln((n)^2)}{8\pi} \leq -\pi((n)^2) \leq -\int_0^{n^2} \frac{dt}{\ln(t)} + \frac{\sqrt{n^2}\ln(n^2)}{8\pi} \end{cases}$$
*So, by summation of the inequalities of the precedent system, we have :
$\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} - \frac{(n+1)\ln(n+1)}{2\pi} \leq \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} - \frac{(n+1)\ln(n+1)}{4\pi} - \frac{n\ln(n)}{4\pi} \leq card(A_n) = \pi((n+1)^2) - \pi(n^2) \leq \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} + \frac{(n+1)\ln(n+1)}{4\pi} + \frac{n\ln(n)}{4\pi} \leq \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} + \frac{(n+1)\ln(n+1)}{2\pi}$
*The result follows.

Lemma4 : we have :
$$\exists p \in \mathbb{N} \forall n \geq p: card(A_n) > \frac{1}{2}\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} \geq \frac{1}{2}\left(\frac{(n+1)^2 - n^2}{2\ln(n+1)}\right) = \frac{2n+1}{4\ln(n+1)} > 0$$

Proof :(of lemma4)

Claim1 : we have : $\frac{2n+1}{2\ln(n+1)} \leq \int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)} \leq \frac{2n+1}{2\ln(n)}$

Proof : (of claim1)
The result follows because :
$$\frac{1}{\ln((n+1)^2)} = \frac{1}{2\ln(n+1)} \leq \frac{1}{\ln(t)} \leq \frac{1}{\ln(n^2)} = \frac{1}{2\ln(n)} \ \forall t \in [n^2, (n+1)^2]$$

Claim2: we have : $\lim_{n \to +\infty} \frac{\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)}}{(n+1)\ln(n+1)} = 0$

Proof : (of claim2)
The result follows by letting: $n \to +\infty$, in the below relation deduced from claim1 :
$$0 \leq \frac{\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)}}{(n+1)\ln(n+1)} \leq \frac{2n+1}{2(n+1)\ln(n+1)\ln(n)}$$
Because : $\lim_{n \to +\infty} \frac{2n+1}{2(n+1)\ln(n+1)\ln(n)} = 0$

Claim3 : we have : $0 \leq liminf\left(\frac{card(A_n)}{\sqrt{n+1}\ln(n+1)}\right) = liminf\left(\frac{O(\sqrt{n+1}\ln(,n+1))}{\sqrt{n+1}\ln(n+1)}\right) \leq \frac{1}{2\pi}$

Proof : (of claim3)
*By lemma2 and the assertion (2) of lemma6, we have :
$liminf\left(\frac{card(A_n)}{\sqrt{n+1}\ln(n+1)}\right) = liminf\left(\frac{\int_{n^2}^{(n+1)^2} \frac{dt}{\ln(t)}}{(n+1)\ln(n+1)} + \frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right)$

$= \lim_{n\to+\infty} \left(\frac{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}{(n+1)\ln(n+1)}\right) + liminf\left(\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right) = 0 + liminf\left(\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right) = liminf\left(\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right)$

*The bounds are obtained by application of the assertion (3) of proposition6, because for any $n \geq \sqrt{2657}$:

$$\frac{card(A_n)}{(n+1)\ln n+1)} \geq 0 \; and \; \frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)} \leq \frac{1}{2\pi} \text{ (By lemma2)}$$

Claim4 : we have :

(1) $liminf\left(\frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) \geq 1$

(2) $\exists p \in \mathbb{N} \forall n \geq p \;\; card(A_n) \geq \frac{1}{2}\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)} > 0$

Proof : (of lemma4)
(1) We have by the assertion (5) of proposition6, lemma2 and claim3:

$$liminf\left(\frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) = liminf\left(1 + \frac{O((n+1)\ln(n+1))}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) = 1 + liminf\left(\frac{O((n+1)\ln(n+1))}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right)$$

$$=1+liminf\left(\left(\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right)\left(\frac{(n+1)\ln(n+1)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right)\right) \geq 1 + liminf\left(\frac{O((n+1)\ln(n+1))}{(n+1)\ln(n+1)}\right)liminf\left(\frac{(n+1)\ln(n+1)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) \geq 1$$

(2) we can consider the two below cases

First case : if $liminf\left(\frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) = +\infty$

*By the assertion (6) of proposition 8, we have : $\lim_{n\to+\infty}\left(\frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) = +\infty,$

*So, by the assertion (1) of proposition 6 (the definition of this limit written for$\epsilon = \frac{1}{2}$): we have :

$$\exists p \in \mathbb{N} \forall n \geq p \;\; \frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}} \geq \frac{1}{2} > 0$$

Second case : if f $liminf\left(\frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}}\right) = s \in \mathbb{R}$

*By the assertion (1) of claim4, we have :$s \geq 1$

*By the assertion (4) of proposition (8), written for$\epsilon = \frac{1}{2}$, we have :

$$\exists p \in \mathbb{N} \forall n \geq p: \frac{card(A_n)}{\int_{n^2}^{(n+1)^2}\frac{dt}{\ln(t)}} > s - \frac{1}{2} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

*This finishes the proof of claim4.

Lemma5 : $\forall n \in \mathbb{N}^* \; card(A_n) > 0$

Proof : (of lemma5)
Consider the subsets :
$A = \{p \in \mathbb{N}, \forall n \geq p: card(A_n) > 0\}$
$B = \{p \in \mathbb{N}, \exists n \geq p: card(A_n) = 0\}$
$C = \{n \in \mathbb{N}, card(A_n) = 0\}$

Claim5 : the subset A has a minimal element: $m = \min(A) \geq 1$, with : $\forall n \geq m: card(A_n) > 0$ and $card(A_{m-1}) = 0$

Proof : (of claim5)
*By the assertion (2) of claim4 : the subset $A \neq \emptyset$, so the result follows by application of the assertion (i) of proposition9.

$*card(A_0) = 0 \Rightarrow m \geq 1$

Claim6 : we have : $B = \mathbb{N} - A$

Proof : (of claim6)
The result follows by definition of A and B.

Claim7 : B is a finite subset with $\max(B) = m - 1$

Proof : (of claim7)
*By claim5, we have : $m - 1 \in B$ and $m \notin B$
*Suppose :$\exists p \in B$ such that :$p > m - 1$, i.e. : $p \geq m$. So :$p \in A$.
*This being contradictory : $\forall p \in B: p \leq m - 1$
*So, by the assertion (ii) of proposition9 : $m - 1 = \max(B)$

Claim8 : $A = \{m, m+1, m+2, \ldots\} = \{p \in \mathbb{N}, p \geq m\}$

Proof : (of claim8)
*we have : $p \in A \Rightarrow \forall n \geq p: card(A_n) > 0 \Rightarrow \forall n \geq p + 1 \geq p: card(A_n) > 0 \Rightarrow p + 1 \in A$
*So : $m = \min(A) \Rightarrow A = \{m, m+1, m+2, \ldots\} = \{p \in \mathbb{N}, p \geq m\}$

Claim9 : we have : $B = \{0, \ldots, m-1\} = \{p \in \mathbb{N}, 0 \leq p \leq m - 1\}$

Proof : (of claim9)
The result is obtained by combination of claim 6 and claim8.

Claim10 : we have : $C \subset B$

Proof : (of claim10)
*We have :$p \in C \Rightarrow card(A_p) = 0 \Rightarrow \exists n = p \geq p: card(A_n) = 0 \Rightarrow p \in B$
*The result follows.

Claim11 : we have : $\max(C) = m - 1$

Proof : (of claim11)
*We have :$card(A_{m-1}) = 0 \Rightarrow m - 1 \in C$ and $card(A_m) > 0 \Rightarrow m \notin C$
*$C \subset B \Rightarrow \forall n \in C: n \leq m - 1$
*So, the assertion (ii) of lemma9 gives the result.

Claim12 : we have : $m = 1$

Proof : (of claim12)
Suppose : $m > 1$
The claim 12 will be deduced from the under-claims below.

Under claim1 : we have :
(1)$p \in A_n \Leftrightarrow [\sqrt{p}] = n$
(2)$m - 1 \in C \Leftrightarrow \forall p \in \mathbb{P}\ |m - 1 - [\sqrt{p}]| > 0$

Proof : (of the under-claim1)
(1) $p \in A_n \Leftrightarrow n^2 < p < (n+1)^2 \Leftrightarrow n < \sqrt{p} < n + 1 \Leftrightarrow [\sqrt{p}] = n$ (by definition of the integer part of a real)
(2) The result follows immediately by use of the definitions.

Under-claim2 : let: $n < m - 1$, $p \in \mathbb{P}$ and $k, r \in \mathbb{N}^*$, we have :

(i)$|n-[\sqrt{p}]|<|m-1-[\sqrt{p}]|$
(ii)$\exists\theta(n,p,k,r)=\alpha_{k,r}\in]\frac{1}{2},\frac{2}{3}-\frac{1}{r}[$ such that :
$\alpha_{k,r}(|n-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-\alpha_{k,r}}=(1-\alpha_{k,r})(|m-1-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-\alpha_{k,r}}$
(iii)$\exists\alpha=\alpha_r\in]\frac{1}{2},\frac{2}{3}-\frac{1}{r}[$ such that :
$\alpha_r(|n-[\sqrt{p}]|)^{\frac{2}{3}-\alpha_r}=(1-\alpha_r)(|m-1-[\sqrt{p}]|)^{\frac{2}{3}-\alpha_r}$
(iv)$\forall p\in\mathbb{P}\ |n-[\sqrt{p}]|>0$
(v)$\forall n\in\{0,1\ldots,m-1\}=B: card(A_n)=0$
(vi)$B=C$

Proof : (of the under-claim2)
(I) We have :
$n<m-1\Rightarrow n(n-2[\sqrt{p}])<(m-1)(m-1-2[\sqrt{p}]$
$\Rightarrow n^2-2n[\sqrt{p}]+([\sqrt{p}])^2<(m-1)^2-2(m-1)[\sqrt{p}]+([\sqrt{p}])^2$
$\Rightarrow (n-[\sqrt{p}])^2<(m-1-[\sqrt{p}])^2\Rightarrow |n-[\sqrt{p}]|<|m-1-[\sqrt{p}]|$
(ii)Consider the continuous function $f_{k,r}$ defined on $[\frac{1}{2},\frac{2}{3}-\frac{1}{r}]$, by :

$$f_{k,r}(t)=t(|n-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-t}-(1-t)(|m-1-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-t}$$

We have :
*By the assertion (i) of the under-claim2 :
$f_{k,r}\left(\frac{1}{2}\right)=\frac{1}{2}(|n-[\sqrt{p}]|+\frac{1}{k})^{\frac{1}{6}}-\frac{1}{2}\left(|m-1-[\sqrt{p}]|+\frac{1}{k}\right)^{\frac{1}{6}}<0$
$f_{k,r}\left(\frac{2}{3}-\frac{1}{r}\right)=(\frac{2}{3}-\frac{1}{r})(|n-[\sqrt{p}]|+\frac{1}{k})^{\frac{1}{r}}-\left(\frac{1}{3}+\frac{1}{r}\right)\left(|m-1-[\sqrt{p}]|+\frac{1}{k}\right)^{\frac{1}{r}}>0$ for a great positive integer r, because by the assertion (2) of proposition6 (written for $\epsilon=\frac{1}{6}$), we have :

$$\lim_{r\to+\infty}f_{k,r}\left(\frac{2}{3}-\frac{1}{r}\right)=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}\Rightarrow\exists N\in\mathbb{N}\forall r\geq N\ f_{k,r}\left(\frac{2}{3}-\frac{1}{r}\right)\geq\frac{1}{3}-\frac{1}{6}=\frac{1}{6}>0$$

*So, by the intermediate value theorem (See proposition5), we have :

$$\exists\alpha_{k,r}\in]\frac{1}{2},\frac{2}{3}-\frac{1}{r}[\text{ Such that : }f_{k,r}(\alpha_{k,r})=0$$

(iii)*By proposition10, the bounded sequence $(\alpha_{k,r})_k$ (for a fixed r and a variable$k$), has a convergent subsequence, denoted also :$(\alpha_{k,r})_r$ such that :

$$\lim_{k\to+\infty}\alpha_{k,r}=\alpha_r=\alpha\in[\frac{1}{2},\frac{2}{3}-\frac{1}{r}]$$

*By proposition7, we have :

$$\lim_{k\to+\infty}f_{k,r}(\alpha_{k,r})=0\Leftrightarrow\alpha(|n-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-\alpha}=(1-\alpha)(|m-1-[\sqrt{p}]|+\frac{1}{k})^{\frac{2}{3}-\alpha}$$

(iv) So, the result follows by combination of the assertions (2) of under-claim1.
(v) and (vi) follows immediately.

Under-claim3 : $m=1$
Proof : (of the under-claim3)
*We have : $1\notin C$
*So, by the assertion (vi) of under-claim2 : $m-1=\max(C)<1\Leftrightarrow m<2\Leftrightarrow m\leq 1$, this being contradictory with our hypothesis : "$m>1$" we have effectively showed that : $m=1$

PROOF OF THE EULER CONJECTURE

Theorem2 : (The Euler conjecture is true) we have, there is an infinite number of prime integers of the form n^2+1.
Proof : (of theorem2)
Let :$\mathbb{E}=\{p\in\mathbb{P},\exists n\in\mathbb{N}\text{ such that }p=n^2+1\}$,
For : $m\in\mathbb{N}^*-\{1\}$, Let : $\mathbb{E}_m=\{p\in\mathbb{P}_m,\exists n\in\mathbb{N},p=n^2+1\}$
For $n\in\mathbb{N}^*$, $\mathbb{P}_{n^2+1}=\{p\in\mathbb{P},p\leq n^2+1\}$

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$\mathbb{F}_n = \{p \in \mathbb{P}, p = n^2 + 1\}$
Euler conjecture is that the subset $\mathbb{E}$ is infinite.
The proof of theorem2 will be deduced from the lemmas below.

Lemma6 : we have :$\forall m \in \mathbb{N}^* - \{1\}\ \mathbb{E}_m \subset \mathbb{E}$

Proof : (of lemma6)
The result follows because : $\forall m \in \mathbb{N}^* - \{1\}: \mathbb{P}_m \subset \mathbb{P}$

Lemma7 : we have :

$$(1) card(\mathbb{F}_n) = \pi(n^2+1) - \pi(n^2) = \begin{cases} 1\ if\ n^2+1 \in \mathbb{P} \\ 0\ if\ n^2+1 \notin \mathbb{P} \end{cases}$$

$$(2) card(\mathbb{E}_m) = \sum_{n=1}^{m} \pi(n^2+1) - \pi(n^2)$$

Proof : (of lemma7)
(1)*We have : $\mathbb{F}_n = \{p \in \mathbb{P}, n^2 < p \le n^2+1\} = \mathbb{P}_{n^2+1} - \mathbb{P}_{n^2}$
*So, by the assertion (1) of proposition1 : $card(\mathbb{F}_n) = card(\mathbb{P}_{n^2+1}) - card(\mathbb{P}_{n^2})$
$= \pi(n^2+1) - \pi(n^2)$
(2)* We have : $\mathbb{E}_m = \bigcup_{n=1}^{m} \mathbb{F}_n$ with $\mathbb{F}_n \cap \mathbb{F}_k = \emptyset\ for\ n \neq k$
*So, by the assertion (3) of proposition1 :
$card(\mathbb{E}_m) = \sum_{n=1}^{m} card(\mathbb{F}_n) = \sum_{n=1}^{m}(\pi(n^2+1) - \pi(n^2))$

Lemma8 : we have for great values of m:

$$-(m-[2657]+1)\frac{\sqrt{m^2+1}\ln(m^2+1)}{8\pi} \le 0 \le \sum_{n=1}^{[\sqrt{2657}]+1}(\pi(n^2+1) - \pi(n^2) \le ([\sqrt{2657}]+1)\pi(([\sqrt{2657}]+1)^2+1) \le (m-[2657]+1)\frac{\sqrt{m^2+1}\ln(m^2+1)}{4\pi}$$

Proof : (of lemma8)
The proof follows immediately.

Lemma9 : we have, for great values of m,

$$\sum_{n=[\sqrt{2657}]}^{m} \int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} - \frac{1}{4\pi}(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1) \le \sum_{n=[\sqrt{2657}]}^{m}(\pi(n^2+1) - \pi(n^2)) \le \sum_{n=[\sqrt{2657}]}^{m} \int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} + \frac{1}{4\pi}(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)$$

Proof : (of lemma9)
*By the Schoenfeld inequality we have, for : $[\sqrt{2657}] \le n \le m$

$$\int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} - \frac{\sqrt{n^2+1}\ln(n^2+1)}{8\pi} \le \pi(n^2+1) - \pi(n^2) \le \int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} + \frac{\sqrt{n^2+1}\ln(n^2+1)}{4\pi}$$

*So, by summing, we have :

$$\int_0^1 \sum_{[\sqrt{2657}]}^{m} \frac{dt}{\ln(t+n^2)} - \frac{1}{4\pi}(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1) \sum_{n=[\sqrt{2657}]}^{m} \int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} - \frac{1}{4\pi}\sum_{n=[\sqrt{2657}]}^{m} \sqrt{n^2+1}\ln(n^2+1) \le \sum_{n=[\sqrt{2657}]}^{m}(\pi(n^2+1) - \pi(n^2)) \le \sum_{n=[\sqrt{2657}]}^{m} \int_{n^2}^{n^2+1} \frac{dt}{\ln(t)} + \frac{1}{4\pi}\sum_{n=[\sqrt{2657}]}^{m} \sqrt{n^2+1}\ln(n^2+1) \le \int_0^1 \sum_{[\sqrt{2657}]}^{m} \frac{dt}{\ln(t+n^2)} + \frac{1}{4\pi}(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)$$

Lemma10 : we have for great values of m:

$$card(\mathbb{E}_m) = \int_0^1 \sum_{[\sqrt{2657}]}^{m} \frac{dt}{\ln(t+n^2)} + O((m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1))$$

With : $\frac{|O((m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1))|}{m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)} \le \frac{1}{2\pi}$

Proof : (of lemma10)

Noting that :$card(\mathbb{E}_m)=\sum_{n=1}^{m}\pi(n^2+1)-\pi(n^2))=\sum_{n=[\sqrt{2657}]}^{m}\left(\pi(n^2+1)-\pi(n^2)\right)+\sum_{n=1}^{[\sqrt{2657}]+1}\left(\pi(n^2+1)-\pi(n^2)\right)$, the result is obtained by combination of lemma8 and lemma9.

Lemma11 : We have : $\lim_{m\to+\infty}\frac{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}=0$

Proof : (of lemma11)
The result follows because : $\ln(([\sqrt{2657}])^2)\leq\ln(t+n^2)\forall t\in[0,1]$ and $n\geq[\sqrt{2657}]$

$$0\leq\frac{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}\leq\frac{1}{\sqrt{m^2+1}\ln(m^2+1)\ln\left(\left([\sqrt{2657}]\right)^2\right)}$$

Lemma12 : we have :
$0\leq liminf\left(\frac{card(\mathbb{E}_m)}{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1}\right)=liminf(\frac{O(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1))}{m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)})\leq\frac{1}{2\pi}$

Proof : (of lemma12)
The result is obtained by use of the assertions(2), (3) of proposition8, lemma10 and lemma11.

Lemma13 : $\lim_{m\to+\infty}\frac{m-[\sqrt{2657}]+1}{\ln(m^2+1)}$=+∞
Proof :(of lemma13)
By the Hôpital rule, we have :

$$\lim_{m\to+\infty}\frac{m-[\sqrt{2657}]+1}{\ln(m^2+1)}=\lim_{m\to+\infty}\frac{1}{\frac{2m}{m^2+1}}=\lim_{m\to+\infty}\frac{m^2+1}{2m}=\lim_{m\to+\infty}m=+\infty$$

Lemma14 : we have : $\lim_{m\to+\infty}\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}$=+∞

Proof : (of lemma13)
*We have : $\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}\geq\frac{(m-[\sqrt{2657}+1]}{\ln(1+m^2)}$
*So, lemma13 gives the result by tending m to $+\infty$

Lemma15 : (1) $liminf\left(\frac{card(\mathbb{E}_m)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}\right)\geq 1$

(2)$\exists p\in\mathbb{N}\forall m\geq p\ \ card(\mathbb{E}_m)\geq\frac{1}{2}\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}$

Proof : (of lemma15)
(1)By combination of the assertions (4), (5) of proposition 8, lemma10 and lemma12, we have successively :

$$liminf\left(\frac{card(\mathbb{E}_m)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}\right)=1+liminf(\frac{O((m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1))}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}})$$

$$=1+liminf(\frac{O((m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1))}{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}\times\frac{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}})$$

$$\geq 1+liminf\left(\frac{O\left((m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)\right)}{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}\right)liminf(\frac{(m-[\sqrt{2657}]+1)\sqrt{m^2+1}\ln(m^2+1)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}))\geq 1$$

(2)***First case :** $liminf\left(\frac{card(\mathbb{E}_m)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}\right)=+\infty$

The result is obtained by writing the definition $\lim_{m\to+\infty}\left(\frac{card(\mathbb{E}_m)}{\int_0^1\sum_{n=[\sqrt{2657}]}^{m}\frac{dt}{\ln(t+n^2)}}\right)=+\infty$ for $\epsilon=\frac{1}{2}$

Second case : $liminf\left(\frac{card(\mathbb{E}_m)}{\int_0^1 \sum_{n=[\sqrt{2657}]}^{m} \frac{dt}{\ln(t+n^2)}}\right) = s \in \mathbb{R}$

The result is obtained by writing, for $\epsilon = \frac{1}{2}$, the first relation in the definition (given in the assertion (4) of proposition8) of : $liminf\left(\frac{card(\mathbb{E}_m)}{\int_0^1 \sum_{n=[\sqrt{2657}]}^{m} \frac{dt}{\ln(t+n^2)}}\right) = s \in \mathbb{R}$, noting by the assertion (1) of lemma15, that : $s \geq 1$.

*This ends the proof of lemma15.

Lemma16 : we have : (1) $\lim_{m\to+\infty} card(\mathbb{E}_m) = +\infty$
(2)$card(\mathbb{E}) = +\infty$

Proof : (of lemma16)

(1) The result follows by combination of lemma14 and the assertion (2) of lemma15.
(2) Suppose contrarily that : $card(\mathbb{E}) < +\infty$. By the assertion (2) of proposition1 and lemma6, we have : $\forall m \in \mathbb{N}^* - \{1\}: \mathbb{E}_m \subset \mathbb{E} \Rightarrow card(\mathbb{E}_m) \leq card(\mathbb{E})$. The contradiction is obtained by tending$m$ to $+\infty$, using the assertion (1) of lemma16.

CONCLUSIONS

My works [5], [6] together with the present paper resolve all the four Landau problems via the Schoenfeld inequality and via some results of elementary analysis.

Remark :I want here to give a little modification to my paper [6] entitled « confirmation of the De Polignac and the Twin primes conjecture » published in the July 2018 issue of the GJAETS:
**In page 3, replace proposition5, by the new proposition 5 :

Proposition5 : for any sequence$(x_n)_n$, we have :
(1)$liminf x_n = sup_{p\in\mathbb{N}} inf_{n\geq p} x_n$ exits always in $[-\infty, +\infty]$
(2)$a \leq x_n \leq b \forall n \Rightarrow a \leq liminf x_n \leq b$
(3)$liminf x_n + liminf y_n \leq liminf(x_n + y_n)$ for any sequence $(y_n)_n$
(4)$y_n \geq 0 \forall n \Rightarrow liminf(x_n y_n) \geq liminf(x_n) liminf(y_n)$
(5)$liminf x_n = +\infty \Leftrightarrow \lim_{n\to+\infty} x_n = +\infty$
(6) if $(y_n)_n$ is a convergent sequence, then :$liminf(x_n + y_n) = liminf(x_n) + \lim_{n\to+\infty} y_n$
**In page 5, replace lemma7 and lemma8, by the new single lemma7 :

Lemma7 : we have :
$$0 \leq liminf\left(\frac{card(J_{n,2k})}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) = liminf\left(\frac{O((\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n))}{(\pi(n)-\pi(2657))\sqrt{2k+n}\ln(2k+n)}\right) \leq \frac{1}{2\pi}$$

Proof : (of lemma7)
*The equality is obtained by combination of the assertion (6) of proposition5, lemma4 and lemma6.
*The first inequality is obtained by application of the assertion (2) of proposition5.
*The second inequality is obtained by application of the assertion (2) of proposition 5 and the last relation of lemma4.
*Replace lemma9, by the new lemma8 :

Lemma8 :$\forall k \in \mathbb{N}^*$, we have : $\lim_{n\to+\infty} card(J_{n,2k}) = +\infty$

Proof :(of lemma8)
*By the assertions (3) and (4) of proposition5 and lemma4 and the first inequality of lemma7, we have successively : $\forall k \in \mathbb{N}^*$
$$liminf(card(J_{n,2k}))=liminf\left(\int_0^1 \sum_{q\in\mathbb{P}_{n,q\geq 2657}} \frac{dt}{\ln(t+2k+q-1)} + O\left(\pi(n)-\pi(2657)\right)\sqrt{n+2k}\ln(n+2k)\right) \geq$$
$$liminf\left(\int_0^1 \sum_{q\in\mathbb{P}_{n,q\geq 2657}} \frac{dt}{\ln(t+2k+q-1)}\right) + liminf\left(O\left(\left(\pi(n)-\pi(2657)\right)\sqrt{n+2k}\ln(n+2k)\right)\right) \geq$$

$liminf\left(\int_0^1 \sum_{q\in\mathbb{P}_{n,q\geq 2657}} \frac{dt}{\ln(t+2k+q-1)}\right) + liminf\left(\frac{O\left((\pi(n)-\pi(2657))\sqrt{n+2k}\ln(n+2k)\right)}{(\pi(n)-\pi(2657))\sqrt{n+2k}\ln(n+2k)}\right) liminf\left((\pi(n)-\pi(2657))\sqrt{n+2k}\ln(n+2k)\right)$

$\geq liminf\left(\int_0^1 \sum_{q\in\mathbb{P}_{n,q\geq 2657}} \frac{dt}{\ln(t+2k+q-1)}\right)$

*The result follows, then, by the assertion (5) of proposition5 and lemma5.
**The rest of the paper [6] remains intact.

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