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A SHORT ELEMENTARY PROOF OF THE BEAL CONJECTURE WITH DEDUCTION OF THE FERMAT LAST THEOREM

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ABSTRACT

The present short paper, which is an amelioration of my previous article "confirmation of the Beal-Brun-Tijdeman-Zagier conjecture" published by the GJETS in 20/11/2019 [15], confirms the Beal's conjecture, remained open since 1914 and saying that:

$$"\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^x + b^y = c^z \text{ and } \gcd(a,b,c) = 1 \Rightarrow \min(x,y,z) = 2"$$

The proof uses elementary tools of mathematics, such as the L'Hôpital rule, the Bolzano-Weierstrass theorem, the intermediate value theorem and the growth properties of certain elementary functions. The proof uses also the Catalan-Mihailescu theorem [18] [19] and some methods developed in my paper on the Fermat last theorem [14] published by the GJAETS in 10/12/2018. The particular case of the Fermat last theorem is deduced.
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KEYWORDS: A Diophantine equation-The Fermat generalized equation-The Fermat/Catalan equation-The Beal conjecture-The Fermat last theorem -Primitive integers-The Intermediate value theorem-The L'Hôpital rule –The Bolzano/Weierstrass theorem -The Catalan/Mihailescu theorem 2010 Mathematics Subject Classification: 11 A xx (Elementary Number theory).

INTRODUCTION

The Beal conjecture:

Definition1: We call « the Tijdeman and Zagier conjecture » or « the Beal conjecture » or what I call « the Beal-Brun-Tijdeman-Zagier conjecture» the following assertion : « the Diophantine equation $a^x + b^y = c^z$ (called « the Fermat generalized equation » or « the Fermat-Catalan equation ») has no solution in $\mathbb{N}^*$ for : $x > 2, y > 2, z > 2$ and $\gcd(a,b,c) = 1$(we say that : a, b, c are primitive)», $\gcd(a,b,c)$ denoting the greatest common divisor of the natural integers a, b and c.

Remark: 1) The case $x = y = z = n$ is the Fermat last theorem that a I have showed in 10/12/2018 (see [14]) by an elementary short proof and also showed in a hard, relatively long, proof in 1994 (see [32]) by A .Wiles.

2) The condition $\gcd(a,b,c) = 1$ is done there to avoid trivialities. Indeed, going from the Diophantine equation: $a^x + b^y = c^z$ multiplied by $a^{21x}b^{14y}c^{6z}$, we obtain an infinite number of solutions of the Diophantine equation: $A^2 + B^3 = C^7$ as: $(a^{11x}b^{7y}c^{3z})^2 + (a^{7x}b^{5y}c^{2z})^3 = (a^{3x}b^{2y}c^z)^7$

Recall that the Diophantine equation $A^2 + B^3 = C^7$ was completely resolved, in [20], where Poonen-Schaefer-Stoll showed that the strictly positive entire primitive solutions are:

$$(A,B,C) = (2213459, 1414, 65) \text{ And } (A,B,C) = (15312283, 9262, 113)$$

3) Recall that if $\zeta(3) = \sum_{k=1}^{+\infty} \frac{1}{k^3}$ denotes the Apéry's constant, the probability for three integers to be primitive is $\frac{1}{\zeta(3)}$

4) Poonen-Schaefer-Stoll remarked in [20] that: for any integers (a,b,c) with $|a| \le K^{\frac{1}{x}}, |b| \le K^{\frac{1}{y}}, |c| \le K^{\frac{1}{z}}$ the probability to have: $a^x + b^y = c^z$ is $1/K$.

History: *the conjecture was formulated independently by the banker and amateur mathematician Andrew Beal [24] in 1993 and the mathematicians Robert Tijdeman and Don Zagier in 1994. But it seems that it has appeared in the Brun works since 1914(see [8]). Andrew Beal [24] devoted, since 1997, an increasing price for any one can prove or disapprove his conjecture.

*In 300 before Jesus Christ, Euclid resolved completely, in ([12], book x), the Diophantine equation $a^2 + b^2 = c^2$ by giving its general solutions (See proposition8 below).

*In the years 1600 Fermat showed that the Diophantine equation: $a^2 + b^4 = c^4$ has no solution.

*Then it was showed that the Diophantine equation: $a^x + b^4 = c^4$ has no solution for any integer x.

*In 1994, Wiles [32] showed by a relatively long proof of 100 pages that the Diophantine equation:$a^n + b^n = c^n$ has no solution with not null integers a, b, c for $n > 2$ by using powerful tools of number theory. This is the Fermat last theorem.

*In 2002, Preda Mihailescu [18], [19] showed that Diophantine equation: $1 + b^y = c^z$ has the sole solution $(b, c, y, z) = (2,3,3,2)$ (Resolving, so, what is known as the « Catalan conjecture»). The proof uses cyclotomic Fields and Galois modules.

*In 2005 Bjorn Poonen, Edward F. Schaefer and Michael Stoll [20] showed that the Diophantine equations:$a^x + b^y = c^z$ with $\{x, y, z\}$ = all permutations of (2,3,7) have only 4 solutions with no power>2.

*In 2009, David Brown [4] studied the case: $(x, y, z) = (2,3,10)$

*In 2009, Michael Bennet, Jordan Ellenberg and Nathan Ng studied [1] the case:$(x, y, z) = (2,4, n)$ for $n \geq 4$.

*In 2014, Samir Siksek and Michael Stoll studied [22] the case $(x, y, z) = (2,3,15)$

*In 2018, M.Ghanim showed in [14] the Fermat last theorem, which is a special case of the Beal conjecture, by an elementary short proof.

For more History see [2] and [3].

The Fermat last theorem:

Definition2: We call « Fermat last theorem », the following statement: « It does not exist natural integers x, y and z such that: $0 < x < y < z$ *and* $x^n + y^n = z^n$, for n a natural integer ≥ 3 ».

History: *This problem has appeared about the fourth century with the Greek mathematician Diophante (325-410) in his work « Arithmetica » [11] (the problem II.VIII, page 85), but the problem $x^2 + y^2 = z^2$, has appeared and was resolved by Euclid, about 300 before J.C, in his famous "Elements" (The Book X) [12]

*About 1621, the French mathematician Pierre Simon de Fermat (1601-1665) writed in the margin of the page 85 of his copy of [11] nears the statement of the famous problem, the following: « J'ai trouvé une merveilleuse démonstration de cette proposition, mais la marge est trop étroite pour la contenir ». This can be translated as:"I have discovered a truly remarkable proof which this margin is too small to contain". But it seems that Fermat has never published his proof. In any case we don't know now this proof.

*In 1670, the proof of the case n=4, by Fermat, was published by his son Samuel.

*On 4 August 1753, L. Euler wrote to Goldbach claiming to prove the Fermat last theorem for n=3, but his proof, published in his book "Algebra" (1770) is incomplete.

*In 1816, the Paris Sciences Academy devoted a gold medal and 3000 F for who can give a proof of the Fermat last theorem. This offer was retaken in 1850.

*In 1825, Dirichlet (1805-1859) and Legendre (1752-1833) proved the case n=5.

*Searching a solution to the Fermat last theorem, Marie-Sophie Germain (1776-1831) discovered his "Sophie Germain theorem"[28] which says that at least one of the positive integers x, y, z such that $x^p + y^p = z^p$,for $p \geq 3$ a prime integer, must be divisible by p^2 if we can find an auxiliary prime q such that:

1) Two none zero consecutive classes modulo q cannot be simultaneously be p-powers

2) p itself cannot be a p-power modulo q

*In 1832, Dirichlet proved the case n=14.

*In 1839, Lame proved the case n=7

*In 1863, the proof of the case n=3, by Gauss, was published.

*In 1908, The Gottingen University and the Wolfskehl Foundation devoted a price of 100.000 Marks for who can give a proof of the Fermat last theorem before 2008.

*In 1952, Harry Vandiver used a Swac Computer to show that the Fermat last theorem is true for n≤ 2000

*Between 1964 and 1994, Jean-Pierre Sere, Yves Hellegouarch and Robert Langlands have given some development to the problem by working on the representation of the elliptic curves with the modular functions.

* This problem, then, remained open for more than 370 years, (Although many attempts of the more eminent mathematicians), when in 1994 the English mathematician Andrew Wiles [32] proved it–by a relatively long proof that has occupied about 100 pages- using powerful tools of number theory, such the Shimura-Taniyama-Weil conjecture, the modular forms, the Galoisian representations…

So the problem of finding a short elementary proof of the Fermat last theorem remained open up to 10/12/2018 when M.Ghanim published in the GJAETS (India) his paper entitled:"confirmation of the Fermat last theorem by an elementary short proof" (See [14]).

*For More and detailed History see in Wikipedia the articles on the Fermat last theorem specially [23] with their references and the Simon Singh good book "Le dernier théorème de Fermat" [21].

The note: The purpose of the present short note is to give a relatively elementary proof of the Beal conjecture based on the intermediate value theorem, the Bolzano-Weierstrass theorem, the L'Hôpital Rule, the Catalan-Mihailescu theorem, the growth properties of certain elementary functions and some methods of [14].

Results: Our main results are:

Theorem1: (proving the Beal's conjecture) let x, y *and* z three integers≥ 2, then:

$$\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^x + b^y = c^z \text{ and } \gcd(a, b, c) = 1 \Rightarrow \min(x, y, z) = 2$$

Theorem2: (proving the Fermat last theorem) let $n \geq 2$, then:

$$\exists a, b, c \in \mathbb{N}^* \text{ such that : } a^n + b^n = c^n \text{ and } \gcd(a, b, c) = 1 \Rightarrow n = 2$$

Methods: The methods used in the paper are as follows.

For theorem1: going from the integers $x, y, z \geq 2$ and $a, b, c \in \mathbb{N}^*$such that $\gcd(a, b, c) = 1$, and $a^x + b^y = c^z$, I show that $n = \min(x, y, z) = 2$. First I show that I can suppose $1 \leq a^x < b^y < c^z$ and secondly that I can distinguish the following 2 cases:

First case:$a = 1$.

In this case $n = \min(x, y, z) = 2$ follows from the Catalan-Mihailescu theorem (Proposition 4).

Second case: $a \geq 2$

*Using the intermediate value theorem, I show that:

$$\forall k \geq 4 \exists \theta_k \in]1-\frac{\pi}{4}, 1[\text{ such that } \left(\frac{a^x}{c^z}\right)^{\theta_k} \left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} + \left(\frac{b^y}{c^z}\right)^{\theta_k} \left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} = 1$$

*Then I show that: $\exists p \geq 4$ such that $e^{\frac{\pi n}{8}\left(\frac{1-\theta_p+\frac{1}{p}}{\tan(1-\theta_p)}\right)} \leq a^x < b^y < c^z$ with $n = \min(x, y, z)$ ($a^x \neq b^y$ assured by $\gcd(a, b, c) = 1$)and $a^x + b^y = c^z$, and I show that: for this $p \geq 4$: we have:

$$2 \leq n = \min(x, y, z) \leq \frac{2\tan(1-\theta_p)}{1-\theta_p} \leq \frac{8}{\pi} = 2.546 \ldots \text{ (because: } 0 < 1-\theta_p \leq \frac{\pi}{4}) \text{ i.e. } n = 2$$

For Theorem 2: The method is a simple deduction from theorem1.

Organization of the paper: The note is organized as follows. The §1 is an introduction giving the necessary definitions and some History. The §2 gives the proof ingredients i.e. the results needed in the proofs of our main results. The §3 gives the proof of the Beal conjecture. The §4 gives the deduction of the Fermat conjecture. The §5 gives some references for further reading.

THE PROOF INGREDIENTS

We will need the following results for showing our main theorem. The other results not needed are cited for information.

Proposition1: (gcd) $\gcd(a, b, c)$ is the strictly positive greatest common divisor of the integers a, b, c. If $\gcd(a, b, c) = 1$ we say that "a, b, c are primitive or coprime" we have:

(i)$d = \gcd(a, b, c) \Rightarrow d > 0$ and d divides the three integers a, b, c

(ii)d divides the three integers a, b, c and $d>0 \Rightarrow d \leq \gcd(a, b, c)$

(iii)d divides the three integers $a, b, c \Rightarrow d$ divides $\gcd(a, b, c)$

(iv) (Bezout theorem) $\gcd(a, b, c) = 1 \Leftrightarrow \exists u; v, w \in \mathbb{Z}\ ua + vb + wc = 1$

(v)$\gcd(a, b, c) = d \Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{N}$ Such that $\begin{cases} a = \alpha d, b = \beta d, c = \gamma d \\ \gcd(\alpha, \beta, \gamma) = 1 \end{cases}$

(vi)* $\gcd(a, b, c) = d \Rightarrow \exists \alpha, \beta, \gamma \in \mathbb{Z}\ ua + vb + wc = d$

*The reciprocal implication is not always true

(vii)$\gcd(a, b, c) = 1 \Leftrightarrow \forall n, m, p \in \mathbb{N}^*\ \gcd(a^n, b^m, c^p) = 1$

(viii)$\gcd(a^n, b^n) = (\gcd(a, b))^n$

Proposition2: (The Gauss theorem) [25] if p is a prime integer, then p is a divisor of the integer $c^z \Rightarrow p$ is a divisor of the integer c. Recall that p is a prime integer if its set of divisors is $\{1, p\}$.

Proposition3: (Euclid (300 before J.C)) ([12], book X) the Diophantine equation:$a^2 + b^2 = c^2$, has the particular solution: $(a, b, c) = (3,4,5)$ and has for general solutions:

$$\begin{cases} a = 2xyz \\ b = x(z^2 - y^2) \\ c = x(z^2 + y^2) \end{cases}$$

With: $(x, y, z) \in \{(p, q, r) \in \mathbb{N}^3$ such that $r > q$ and p, q, r are of different parity$\}$

Proposition4 :(Eugene Charles Catalan-Preda Mihailescu theorem) [18], [19] The Diophantine equation: $1 + b^y = c^z$ (with: b, c, z, y integers > 1) has the single solution: $b = 2, c = 3, y = 3$ and $z = 2$.

Proposition5: For: (x, y, z) =all the permutations of {2, 4, 4 }, the Diophantine equation:$a^x + b^y = c^z$ has no solution in $\mathbb{N}^*$.

Proof: (of proposition2)

*For example considering The Diophantine equation $a^2 + b^4 = c^4$ we have:

$$a^2 + (b^2)^2 = (c^2)^2$$

*So, by proposition3: $a = 3, b^2 = 4$ and $c^2 = 5$ is a solution i.e. $a = 3, b = 2$ and $c = \sqrt{5}$

*But $\sqrt{5} \notin \mathbb{Q} \Rightarrow c \notin \mathbb{N}$

*This being impossible the result is showed.

Proposition6: (Euler-Gauss theorem [13], [16], [17]) The Diophantine $a^3 + b^3 = c^3$ has no solutions in $\mathbb{N}^*$.

Proposition7: (The Fermat last theorem) [14], [32], we have:

$$\exists a, b, c \in \mathbb{N}^* \text{ such that } a^n + b^n = c^n \text{ And } \gcd(a, b, c) = 1 \Rightarrow n = 2$$

Proposition8 :(Poonen-Schaefer-Stoll [20]) for $(x, y, z) =$ all the permutations of {2, 3, 6} the sole case for which the Diophantine equation $a^x + b^y = c^z$ has non trivial solutions is $x = 6, a = 1, y = 3, b = 2, z = 2, c = 3$

Proposition9: (Beukers theorem [3]) in the case $(x, y, z) = (2,2,z)$ with:$z \geq 2$ or $(x, y, z) = (2,3,3), (2,3,4), (2,3,5)$), the set of solutions of the Diophantine equation $a^x + b^y = c^z$ is empty or infinite.

Proposition10: (Darmon-Granville theorem [10]) for any fixed choice of positive integers x, y, z satisfying the hyperbolic case:$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, only finitely many primitive triples (a, b, c) solving the Diophantine equation $a^x + b^y = c^z$ exist.

Note that this result resolves partially the Fermat-Catalan conjecture which is stronger because allows the exponents x, y, z to vary.

Proposition11: For the Diophantine equation: $a^x + b^y = c^z$, we can suppose $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$ called the hyperbolic case.

Proof: (of proposition11)

Remark: We have:$\forall x, y, z \in \mathbb{N}^*$ $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{41}{42}$

Indeed for the two other cases, we have:

***Second case called the Euclidean case:**$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, a simple analysis shows that:

$$(x, y, z) = (3,3,3) \text{ or all the permutations of } \{2,4,4\} \text{ or all the permutations of } \{2,3,6\}$$

These cases are completely resolved respectively by proposition6, proposition5 and proposition8.

***Third case called the spherical case:** $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1$, a simple analysis shows that:

$(x, y, z) =$ all the permutations of $\{2,2,\mathrm{m}\} (\mathrm{m} \geq 2)$

or all the permutations of {2,3,3}or all the permutations of {2,3,4}

or all the permutations of {2,3,5}

These cases are completely resolved by the Beukers theorem (proposition9).

Proposition12: From the solutions, (a,b,c,x,y,z), of the Diophantine equation $a^x+b^y=c^z$ with $\gcd(a,b,c)=1$; $x,y,z\geq 2$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<1$, we know the following ten ones :

*$1^6+2^3=3^2, 2^5+7^2=3^4$(N. Bruin, 2003[6]), $13^2+7^3=2^9$(N. Bruin, 2004[7]), $2^7+17^3=71^2$(Poonen – Schaefer – Stoll, 2005[20]), $3^5+11^4=122^2$(Bruin, 2003[6])

*$17^7+76271^3=21063928^2, 1414^3+2213459^2=65^7, 9262^3+15312283^2=113^7$

(The three discovered by Poonen-Schaefer-Stoll, 2005[20])

$43^8+96222^3=30042907^2$(Bruin, 2003[6]), $33^8+1549034^2=15613^3$(Bruin, 1999[5])

Remark: 1) for all the examples of the precedent proposition12, we have:$\min(x,y,z)=2$.

2) S.Siksek and M.Stoll [22] talk, following H.Darmon [9] and H .Darmon-A. Granville [10], about the generalized Fermat conjecture (concerning the hyperbolic case: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<1$) which says that the sole non trivial primitive solutions are those cited in the above proposition12.

Proposition13: the hypothesis « $\gcd(a,b,c)=1$ » is a necessary condition in the Beal conjecture.

Proof: (of proposition13)

(i)For example:

*$2^n+2^n=2^{n+1}$, for:$n\geq 0$. Note that: 2 divides $\gcd(2^n,2^n,2^{n+1})\neq 1$.

*$3^{3n}+(2.3^n)^3=3^{3n+2}$ for $n\geq 1$.

Note that: 3 is a common factor to $a=3, b=2.3^n$ and $c=3$ so $\gcd(a,b,c)\neq 1$.

*$(p^n-1)^{2n}+(p^n-1)^{2n+1}=(p.(p^n-1)^2)^n$ for $n\geq 3$ and $p\geq 2$. Note that a^n-1 is a common factor to $a=p^n-1, b=p^n-1$ and $c=p(p^n-1)^2$ so $gcd(a,b,c)\neq 1$

*$(p(p^n+q^n))^n+(q(p^n+q^n))^n=(p^n+q^n)^{n+1}$ for $n\geq 3$ and $p,q\geq 1$.

Note that: p^n+q^n is a common factor to $a=p(p^n+q^n), b=q(p^n+q^n)$ and $c=p^n+q^n$.

(ii) We can, in fact, construct from any solution (a_1,b_1,c_1) such that $a_1^x+b_1^y=c_1^z$ an infinite number of solutions (a_n,b_n,c_n) such that:

1) $a_n^x+b_n^y=c_n^z$.

2) $a_n=a_{n-1}^{yz+1}.b_{n-1}^{yz}.c_{n-1}^{yz}, b_n=a_{n-1}^{xz}.b_{n-1}^{xz+1}.c_{n-1}^{xz}$ And $c_n=a_{n-1}^{xy}.b_{n-1}^{xy}.c_{n-1}^{xy+1}$

3) $\gcd(a_n,b_n,c_n)\neq 1$ because $a_{n-1}.b_{n-1}.c_{n-1}$ divides $\gcd(a_n,b_n,c_n)$

Proposition14: (The intermediate value theorem) [26] Let $\varphi:[a,b]\to\mathbb{R}$ (with: $a<b$) a continuous function, then :$\varphi(a)\varphi(b)<0 \Rightarrow \exists c\in]a,b[$ such that $\varphi(c)=0$

Proposition15: (circular functions) [31] we have:

(0)$\tan(0) = 0$ and $\tan\left(\frac{\pi}{4}\right) = 1$ with $\tan(t) = \frac{\sin(t)}{\cos(t)}$

(1)$\forall t \in [0,\frac{\pi}{2}]$ $1 \geq \cos(t) \geq 0$ And $1 \geq \sin(t) \geq 0$

(3)The function $t \to \sin(t)$ is increasing on $[0,\frac{\pi}{2}]$ with$(\sin(t))' = \cos(t)$.

(4) The function $t \to \cos(t)$ is decreasing on $[0,\frac{\pi}{2}]$ with$(\cos(t))' = -\sin(t)$.

(5)The function $t \to \tan(t)$ is derivable on $]0,\frac{\pi}{2}[$ with $(\tan(t))' = \left(1 + (\tan(t))^2\right) = \frac{1}{(\cos(t))^2} > 0$ and has a reciprocal function denoted "arctan": $[0,+\infty[\to [0,\frac{\pi}{2}[$

Proposition16: (The l'Hôpital rule) (See [30])

(i) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty, \lim_{x\to a} g(x) = \pm\infty$ (a can be infinite) the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called to be an indeterminate form (IF)$\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively.

(ii) If f, g are differentiable on an interval$]a,b[$ except perhaps in a point $c \in]a,b[$, if $\lim_{x\to c}\frac{f(x)}{g(x)}$ is the IF $\frac{0}{0}$ and if $\forall x \neq c, g'(x) \neq 0$, then: $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ when the limits have a sense.

(iii) If f', g' satisfies the same conditions as f and g the process is repeated.

(iv)The result remain true in the case where $\lim_{x\to c}\frac{f(x)}{g(x)}$ is the IF $\frac{\infty}{\infty}$.

Proposition17: (The Bolzano-Weierstrass theorem) [29] any bounded sequence $(\theta_k)_k \subset]a,b[$ has a subsequence, denoted also $(\theta_k)_k$, converging to $\theta \in [a,b]$

PROOF OF THE BEAL CONJECTURE

THEOREM1: (Beal Conjecture) Let $x,y,z \in \mathbb{N}^* \geq 2$, $(a,b,c) \in \mathbb{N}_*^3$, then:

$$\begin{cases} \gcd(a,b,,c) = 1 \\ a^x + b^y = c^z \\ abc \neq 0 \end{cases} \Rightarrow n = \min(x,y,z) = 2$$

Proof: (of the theorem1)

*Let $x,y,z \in \mathbb{N}^* \geq 2$, $(a,b,c) \in \mathbb{N}_*^3$ such that: $a^x + b^y = c^z$ and gcd (a,b,c) =1,

* Prove that $n = \min(x,y,z) = 2$

Lemma1: $\begin{cases} \gcd(a,b,,c) = 1 \\ a^x + b^y = c^z \\ abc \neq 0 \end{cases} \Rightarrow$ we can suppose $1 \leq a^x < b^y < c^z$

Proof :(of lemma 1)

*$a^x = c^z - b^y > 0 \Rightarrow c^z > b^y$

*the order "$\leq$ " being total on $\mathbb{N}$, we have: $a^x \leq b^y$ *or* $b^y \leq a^x$

*So, we can suppose $a^x \leq b^y$

*I say that: $\gcd(a,b,c)=1 \Rightarrow a^x \neq b^y$

*Indeed if: $a^x = b^y$, we have:$2a^x = c^z$.

*So, 2 being prime, the Gauss theorem (see proposition 2) implies that $c = 2^w l$ where l is not dividable by 2.

*So: $2a^x = 2^{zw} l^z$ or $a^x = 2^{zw-1} l^z$

*But: $w \geq 1, z \geq 2 \Rightarrow wz \geq 2 \Rightarrow wz - 1 \geq 1 \Rightarrow 2$ divides a and b

*So: 2 divides $\gcd(a,b,c)=1$ (see the assertion (iii) of proposition 1)

*This being impossible the result follows.

Lemma2: if: $a = 1$, we have:$n = \min(x,y,z) = 2$

Proof: (of lemma2)

The result follows from the Mihailescu theorem (see proposition 4) assuring that the single solution of the Diophantine equation $1^x + b^y = c^z$ is $(b,y,c,z) = (2{,}3{,}3{,}2)$.

Lemma3: So we can suppose $a \geq 2$

Proof: (lemma3)

The result follows from lemma2, because we work with integers.

Lemma4: We have: (1)$\lim_{t\to 0} \frac{t}{\tan(t)} = 1$

(2)$\forall t \in \left[0, \frac{\pi}{2}\right]\ 0 \leq \frac{t}{\tan(t)} \leq 1$

(3)$\forall t \in \left[0, \frac{\pi}{4}\right]\ 1 \leq \frac{\tan(t)}{t} \leq \frac{4}{\pi}$

Proof: (of lemma4)

(1)*By the L'Hôpital rule, we have successively:

$$\lim_{t\to 0} \frac{\tan(t)}{t} = FI\frac{0}{0} = \lim_{t\to 0} \frac{(\tan(t))'}{t'} = \lim_{t\to 0} \frac{1+(\tan(t))^2}{1} = \lim_{t\to 0}(1 + (\tan(t))^2) = 1$$

(2)*For $f(t) = \tan(t) - t$, we have: $f'(t) = 1 + (\tan(t))^2 - 1 = (\tan(t))^2 \geq 0\ \forall t \in [0, \frac{\pi}{2}] \Rightarrow f$ is increasing on $[0, \frac{\pi}{2}] \Rightarrow \forall t \in \left[0, \frac{\pi}{2}\right] f(t) = \tan(t) - t \geq f(0) = 0$

*The result follows.

(3)*For $g(t) = \frac{4}{\pi}t - \tan(t)$, we have: $g'(t) = \frac{4}{\pi} - ((\tan(t))^2 + 1) = \frac{4}{\pi} - 1 - (\tan(t))^2$

*$g'(t) = 0, t \in \left[0, \frac{\pi}{4}\right] \Leftrightarrow t = \alpha = \arctan\left(\sqrt{\frac{4}{\pi} - 1}\right)$

*g is increasing on $[0, \arctan(\sqrt{\frac{4}{\pi} - 1})]$ and decreasing on $[\arctan\left(\sqrt{\frac{4}{\pi} - 1}\right), \frac{\pi}{4}]$

*So, we have: $0 \leq t \leq \arctan\left(\sqrt{\frac{4}{\pi} - 1}\right) \Rightarrow g(t) = \frac{4}{\pi}t - \tan(t) \geq g(0) = 0$

And: $\arctan\left(\sqrt{\frac{4}{\pi}-1}\right) \le t \le \frac{\pi}{4} \Rightarrow g(t) = \frac{4}{\pi}t - \tan(t) \ge g\left(\frac{\pi}{4}\right) = 0$

*The result follows.

Lemma5: For $k \ge 4$, we have:

(i)The function $v(t) = t^{-\frac{\pi}{4}}(\ln(t))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}$ is decreasing for t$> e^{\frac{\pi}{4}+\frac{1}{k}} > 1$

(ii) $a^x > e^{\frac{\pi}{4}+\frac{1}{k}}$

(iii) $(a^x)^{-\frac{\pi}{4}}(\ln(a^x))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} > (b^y)^{-\frac{\pi}{4}}(\ln(b^y))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} > (c^z)^{-\frac{\pi}{4}}(\ln(c^z))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}$

Proof: (of lemma5)

(i)*We have:

$v'(t) = t^{-\frac{\pi}{4}-1}(\ln(t))^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})-1}\ln\left(t^{-\frac{\pi}{4}}e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right)$

*So: v deceasing $\Leftrightarrow \ln\left(t^{-\frac{\pi}{4}}e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})}\right) \le 0$

$\Leftrightarrow t^{-\frac{\pi}{4}}e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} \le 1 \Leftrightarrow t^{\frac{\pi}{4}} \ge e^{\frac{\pi}{4}(\frac{\pi}{4}+\frac{1}{k})} \Leftrightarrow t \ge e^{\frac{\pi}{4}+\frac{1}{k}}$

(ii)*Suppose contrarily that: $a^x \le e^{\frac{\pi}{4}+\frac{1}{k}}$

* We have:

$$k \ge 4 \Rightarrow 2^2 = 4 \le a^x \le e^{\frac{\pi}{4}+\frac{1}{k}} \le e^{\frac{\pi}{4}+\frac{1}{4}} = 2.816\ldots$$

*This being impossible the result follows.

(iii)The result follows by combination of lemma1 and the assertions (i), (ii) of lemma5.

Lemma6: (i)$\forall k \ge 4 \exists \theta_k \in]1-\frac{\pi}{4}, 1[$ such that:

$$\left(\frac{a^x}{c^z}\right)^{\theta_k}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} + \left(\frac{b^y}{c^z}\right)^{\theta_k}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} = 1$$

(ii) The sequence $(\theta_k)_k$ has a subsequence denoted also $(\theta_k)_k$, converging to $\theta \in [1-\frac{\pi}{4}, 1]$

Proof: (of lemma 6)

(i)*Consider on$\left[1-\frac{\pi}{4}, 1\right]$, for $k \ge 4$, the continuous function:

$$\varphi_k(t) = \left(\frac{a^x}{c^z}\right)^t\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-t+\frac{1}{k})} + \left(\frac{b^y}{c^z}\right)^t\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} - 1$$

*We have:

$$\begin{cases} 4 \le a^x < b^y < c^z \\ \frac{a^x}{c^z} + \frac{b^y}{c^z} = 1 \\ \text{the assertion (iii) of lemma 5} \end{cases} \Rightarrow$$

$$\Rightarrow \varphi_k\left(1-\frac{\pi}{4}\right)=\left(\frac{a^x}{c^z}\right)^{1-\frac{\pi}{4}}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}+\left(\frac{b^y}{c^z}\right)^{1-\frac{\pi}{4}}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-1$$

$$=\left(\frac{a^x}{c^z}\right)^{1-\frac{\pi}{4}}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}+\left(\frac{b^y}{c^z}\right)^{1-\frac{\pi}{4}}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-\left(\frac{a^x}{c^z}+\frac{b^y}{c^z}\right)$$

$$=\frac{a^x}{c^z}\left(\left(\frac{a^x}{c^z}\right)^{-\frac{\pi}{4}}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-1\right)+\frac{b^y}{c^z}\left(\left(\frac{b^y}{c^z}\right)^{-\frac{\pi}{4}}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-1\right)$$

$$=\frac{a^x}{c^z}\left(\frac{(a^x)^{-\frac{\pi}{4}}(\ln(a^x))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-(c^z)^{-\frac{\pi}{4}}(\ln(c^z))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}}{(c^z)^{-\frac{\pi}{4}}(\ln(c^z))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}}\right)+\frac{b^y}{c^z}\left(\frac{(b^y)^{-\frac{\pi}{4}}(\ln(b^y))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}-(c^z)^{-\frac{\pi}{4}}(\ln(c^z))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}}{(c^z)^{-\frac{\pi}{4}}(\ln(c^z))^{\frac{\pi}{4}\left(\frac{\pi}{4}+\frac{1}{k}\right)}}\right)>0$$

* We have:

$$\begin{cases}\frac{a^x}{c^z}+\frac{b^y}{c^z}=1\\ \ln(4)\le\ln(a^x)<\ln(b^y)<\ln(c^z)\Rightarrow\\ \frac{1}{k}>0\end{cases}$$

$$\varphi_k(1)=\frac{a^x}{c^z}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}+\frac{b^y}{c^z}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}-1$$

$$=\frac{a^x}{c^z}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}+\frac{b^y}{c^z}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}-\left(\frac{a^x}{c^z}+\frac{b^y}{c^z}\right)$$

$$=\frac{a^x}{c^z}\left(\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}-1\right)+\frac{b^y}{c^z}\left(\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(\frac{1}{k}\right)}-1\right)<0$$

*So, by the intermediate value theorem: $\varphi_k\left(1-\frac{\pi}{4}\right)\varphi_k(1)<0\Rightarrow\forall k\ge\frac{4}{\pi}\exists\theta_k\in]1-\frac{\pi}{4},1[$ Such that:

$$\varphi_k(\theta_k)=\left(\frac{a^x}{c^z}\right)^{\theta_k}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}+\left(\frac{b^y}{c^z}\right)^{\theta_k}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}-1=0$$

(ii)The result follows by the Bolzano-Weierstrass theorem.

Lemma7: the condition "$a^x\ne b^y$"(assured by the hypothesis "$\gcd(a,b,c)=1$") is necessary for having the assertion (i) of lemma6.

Proof: (of lemma7)

Lemma7 will be deduced from the claims below.

Claim1: We have: $a^x=b^y\Rightarrow a^x=2^{\frac{1}{2^{\frac{1-\theta_k}{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}}-1}}$

Proof: (of claim1)

*Indeed: letting $a^x=b^y$ in the relation equation of the assertion (i) of lemma6, we have successively:

$$\frac{a^x}{c^z}=\frac{1}{2}\Rightarrow 2\left(\frac{1}{2}\right)^{\theta_k}\left(\frac{\ln(a^x)}{\ln(2a^x)}\right)^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}=2^{1-\theta_k}\left(\frac{\ln(a^x)}{\ln(2a^x)}\right)^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}=1$$

$$\Rightarrow\frac{\ln(2)+\ln(a^x)}{\ln(a^x)}=2^{\frac{1-\theta_k}{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}}\Rightarrow\left(2^{\frac{1-\theta_k}{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}}-1\right)\ln(a^x)=\ln(2)\Rightarrow a^x=2^{\frac{1}{2^{\frac{1-\theta_k}{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}}-1}}$$

Claim2: $2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}-1}} \leq 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\tan\left(1-\theta_k+\frac{1}{k}\right)}-1}}$

Proof: (of claim2)

*By the assertion (2) of lemma4, we have:

$k \geq 4 \geq 1.75 \ldots = \frac{1}{\frac{\pi}{2}-1} \Rightarrow 1-\frac{\pi}{2}+\frac{1}{k} < 0 < \theta_k \Rightarrow 1-\theta_k+\frac{1}{k} < \frac{\pi}{2}$

$\Rightarrow 1-\theta_k+\frac{1}{k} \leq \tan\left(1-\theta_k+\frac{1}{k}\right) \Rightarrow \frac{1-\theta_k}{\tan\left(1-\theta_k+\frac{1}{k}\right)} \leq \frac{1-\theta_k}{1-\theta_k+\frac{1}{k}}$

$\Rightarrow 2^{\frac{\frac{4}{\pi}(1-\theta_k)}{\tan\left(1-\theta_k+\frac{1}{k}\right)}}-1 \leq 2^{\frac{\frac{4}{\pi}(1-\theta_k)}{1-\theta_k+\frac{1}{k}}}-1 \Rightarrow \frac{1}{2^{\frac{\frac{4}{\pi}(1-\theta_k)}{1-\theta_k+\frac{1}{k}}}-1} \leq \frac{1}{2^{\frac{\frac{4}{\pi}(1-\theta_k)}{\tan\left(1-\theta_k+\frac{1}{k}\right)}}-1}$

$\Rightarrow 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}-1}} \leq 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\tan\left(1-\theta_k+\frac{1}{k}\right)}-1}}$

Conclusion: (The wanted contradiction)

***First case:** if $\lim_{k\to+\infty} \theta_k = \theta \neq 1$

**By claim1, tending $k \to +\infty$, we have:

$$4 \leq a^x = \lim_{k\to+\infty} 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}-1}} = 2^{\frac{\frac{1}{1-\theta}}{2^{\frac{\pi}{4}(1-\theta)}-1}} = 2^{\frac{1}{2^{\frac{4}{\pi}}-1}} = 1.63 \ldots$$

**This being impossible, the first case cannot occur.

***Second case:** if $\lim_{k\to+\infty} \theta_k = \theta = 1$

**By the assertion (1) of lemma4, claim1 and claim2, we have:

$$4 \leq a^x \leq 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\tan\left(1-\theta_k+\frac{1}{k}\right)}-1}} \Rightarrow 4 \leq a^x \leq \lim_{k\to+\infty} 2^{\frac{\frac{1}{1-\theta_k}}{2^{\frac{\pi}{4}\tan\left(1-\theta_k+\frac{1}{k}\right)}-1}} = 2^{\frac{1}{2^{\lim_{t\to 0}\frac{\frac{4}{\pi}t}{\tan(t)}}-1}} = 2^{\frac{1}{2^{\frac{4}{\pi}}-1}} = 1.63 \ldots$$

**This being impossible, the second case cannot, also, occur.

*Because the two possible cases cannot both occur: lemma 7 follows.

Lemma8: if $n = \min(x, y, z)$ and if θ_k (for $k \geq 4$) is that given by lemma 6, we have:

(i) The function $w(t) = t^{-\frac{2}{n}\tan(1-\theta_k)}(\ln(t))^{\frac{\pi}{4}\left(1-\theta_k+\frac{1}{k}\right)}$ is decreasing for $t \geq e^{\frac{\pi n}{8}\left(\frac{1-\theta_k+\frac{1}{k}}{\tan(1-\theta_k)}\right)} > 1$

(ii)$\exists p \geq 4$ such that $a^x \geq e^{\frac{\pi n}{8}\left(\frac{1-\theta_p+\frac{1}{p}}{\tan(1-\theta_p)}\right)}$

(iii)For $p \geq 4$ given by the assertion (ii)of lemma8, we have:

$(a^x)^{-\frac{2}{n}\tan(1-\theta_p)}(\ln(a^x))^{\frac{\pi}{4}\left(1-\theta_p+\frac{1}{p}\right)} > (b^y)^{-\frac{2}{n}\tan(1-\theta_p)}(\ln(b^y))^{\frac{\pi}{4}\left(1-\theta_p+\frac{1}{p}\right)}$

$> (c^z)^{-\frac{2}{n}\tan(1-\theta_p)}(\ln(c^z))^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})}$

Proof: (of lemma8)

(i)*For $t > 1$, We have:

$$w'(t) = t^{-\frac{2}{n}\tan(1-\theta_k)-1}(\ln(t))^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})-1}\ln\left(t^{-\frac{2}{n}\tan(1-\theta_k)}e^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})}\right)$$

*So, w decreasing$\Leftrightarrow$ $\ln\left(t^{-\frac{2}{n}\tan(1-\theta_k)}e^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})}\right) \le 0 \Leftrightarrow t^{-\frac{2}{n}\tan(1-\theta_k)}e^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} \le 1 \Leftrightarrow t^{\frac{2}{n}\tan(1-\theta_k)} \ge e^{\frac{\pi}{4}(1-\theta_k+\frac{1}{k})} \Leftrightarrow t \ge e^{\frac{\pi n}{8}\left(\frac{1-\theta_k+\frac{1}{k}}{\tan(1-\theta_k)}\right)}$

(ii) *Suppose contrarily that:$\forall k \ge 4$ $a^x < e^{\frac{\pi n}{8}\left(\frac{1-\theta_k+\frac{1}{k}}{\tan(1-\theta_k)}\right)}$

*Tending $k \to +\infty$, we have, by the assertion (ii) of lemma4 and the assertion (ii) of lemma6: $\theta \in [1-\frac{\pi}{4}, 1] \Rightarrow 0 < \frac{1-\theta}{\tan(1-\theta)} \le 1 \Rightarrow a^x \le \lim_{k\to+\infty} e^{\frac{\pi n}{8}\left(\frac{1-\theta_k+\frac{1}{k}}{\tan(1-\theta_k)}\right)} = e^{\lim_{k\to+\infty}\frac{\pi n}{8}\left(\frac{1-\theta_k+\frac{1}{k}}{\tan(1-\theta_k)}\right)} = e^{\frac{\pi n}{8}\left(\frac{1-\theta}{\tan(1-\theta)}\right)} \le e^{\frac{\pi}{8}n}$

*We have: $\frac{n}{x} = \frac{\min(x,y,z)}{x} \le 1 \Rightarrow 2 \le a \le e^{\frac{\pi n}{8x}} \le e^{\frac{\pi}{8}} = 1.48\ldots$

*This being impossible the result follows.

(iii) The result is obtained by combination of lemma1 and the assertions (i) and (ii) of lemma 8.

Lemma9: If p, θ_p are the numbers given respectively by the assertion (iii) of lemma8 and the assertion (i) of lemma6, we have: $\theta_p \ge 1 - \frac{2}{n}\tan(1-\theta_p)$, that is: $n \le \frac{2\tan(1-\theta_p)}{1-\theta_p}$

Proof: (of lemma 9)

***Remark:** we have:

$$\begin{cases} \theta_p \in]1-\frac{\pi}{4}, 1[\\ p \ge 4 \\ n \ge 2 \end{cases} \Rightarrow 0 < \frac{2}{n}\tan(1-\theta_p) < \tan\left(\frac{\pi}{4}\right) = 1$$

* If not: $\theta_p < 1 - \frac{2}{n}\tan(1-\theta_p)$

* So, by the assertion (iii) of lemma7, we have:

$$0 = \left(\frac{a^x}{c^z}\right)^{\theta_p}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} + \left(\frac{b^y}{c^z}\right)^{\theta_p}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} - 1$$

$$> \left(\frac{a^x}{c^z}\right)^{1-\frac{2}{n}\tan(1-\theta_p)}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} + \left(\frac{b^y}{c^z}\right)^{1-\frac{2}{n}\tan(1-\theta_p+\frac{1}{p})}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} - 1$$

$$= \left(\frac{a^x}{c^z}\right)^{1-\frac{2}{n}\tan(1-\theta_p)}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} + \left(\frac{b^y}{c^z}\right)^{1-\frac{2}{n}\tan(1-\theta_p)}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} - \left(\frac{a^x}{c^z}+\frac{b^y}{c^z}\right)$$

$$= \frac{a^x}{c^z}\left(\left(\frac{a^x}{c^z}\right)^{-\frac{2}{n}\tan(1-\theta_p)}\left(\frac{\ln(a^x)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} - 1\right) + \frac{b^y}{c^z}\left(\left(\frac{b^y}{c^z}\right)^{-\frac{2}{n}\tan(1-\theta_p)}\left(\frac{\ln(b^y)}{\ln(c^z)}\right)^{\frac{\pi}{4}(1-\theta_p+\frac{1}{p})} - 1\right)$$

$> 0.$

*The obtained relation "0<0" being impossible, we have well: $\theta_p \geq 1 - \frac{2}{n}\tan(1-\theta_p)$ (That is: $n \leq \frac{2\tan(1-\theta_p)}{1-\theta_p}$)

RETURN TO THE PROOF OF THEOREM1:

*By lemma 9 and the assertion (3) of lemma4, we have:

$$0 < 1 - \theta_p < \frac{\pi}{4} \Rightarrow 2 \leq n \leq \frac{2\tan(1-\theta_p)}{1-\theta_p} \leq 2\left(\frac{4}{\pi}\right) = \frac{8}{\pi} = 2.546.. \Rightarrow n = 2$$

*This ends the proof of the Beal conjecture.

DEDUCTION OF THE FERMAT LAST THEOREM

Theorem2: (the Fermat last theorem) let n an integer≥ 2, then:

$$\exists a,b,c \in \mathbb{N}^* \text{ such that : } a^n + b^n = c^n \text{ and } \gcd(a,b,c) = 1 \Rightarrow n = 2$$

Proof: (of theorem2)

The result follows because the Fermat last theorem is the particular case: $x = y = z = n$ of the Beal conjecture.

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